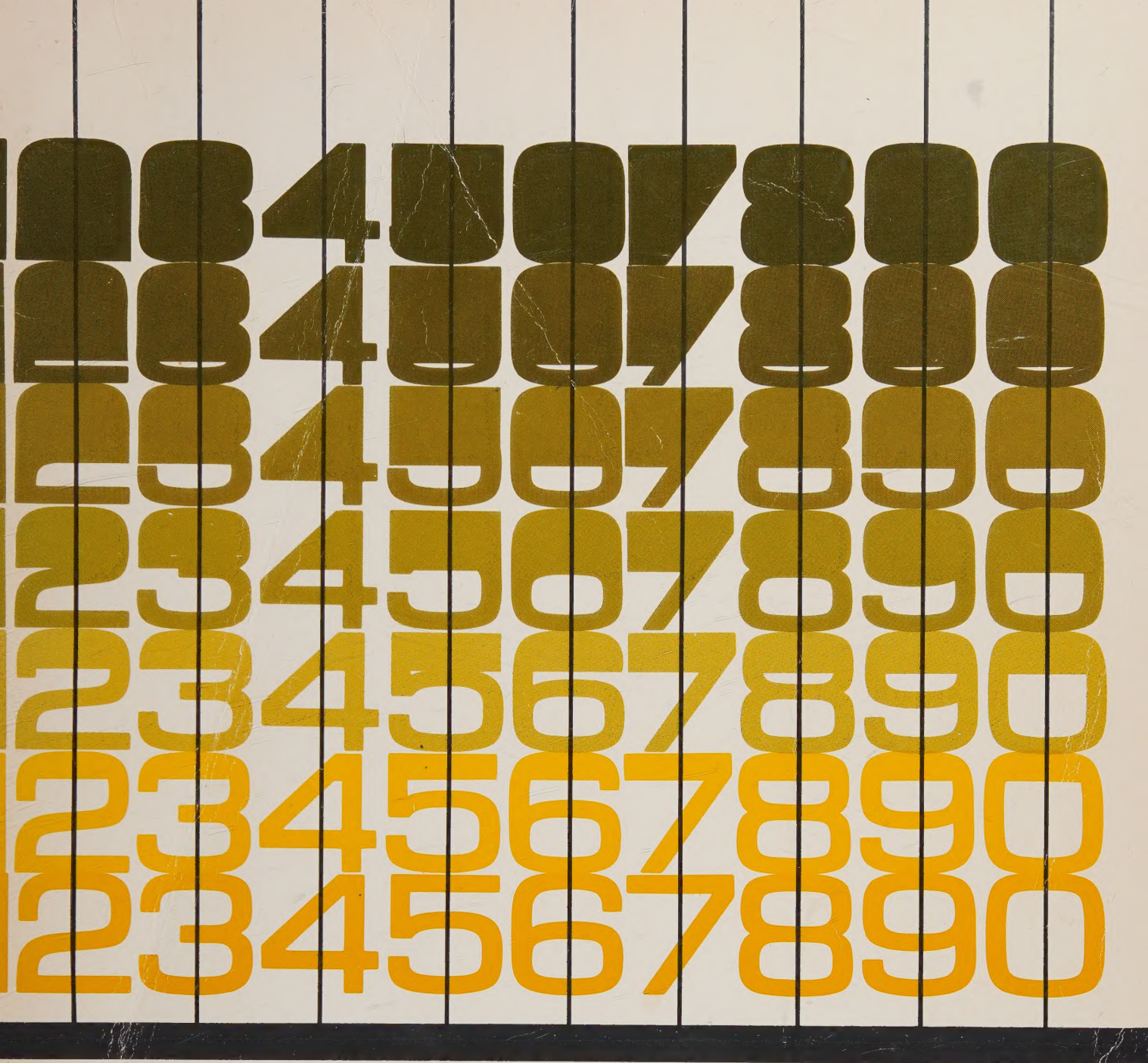
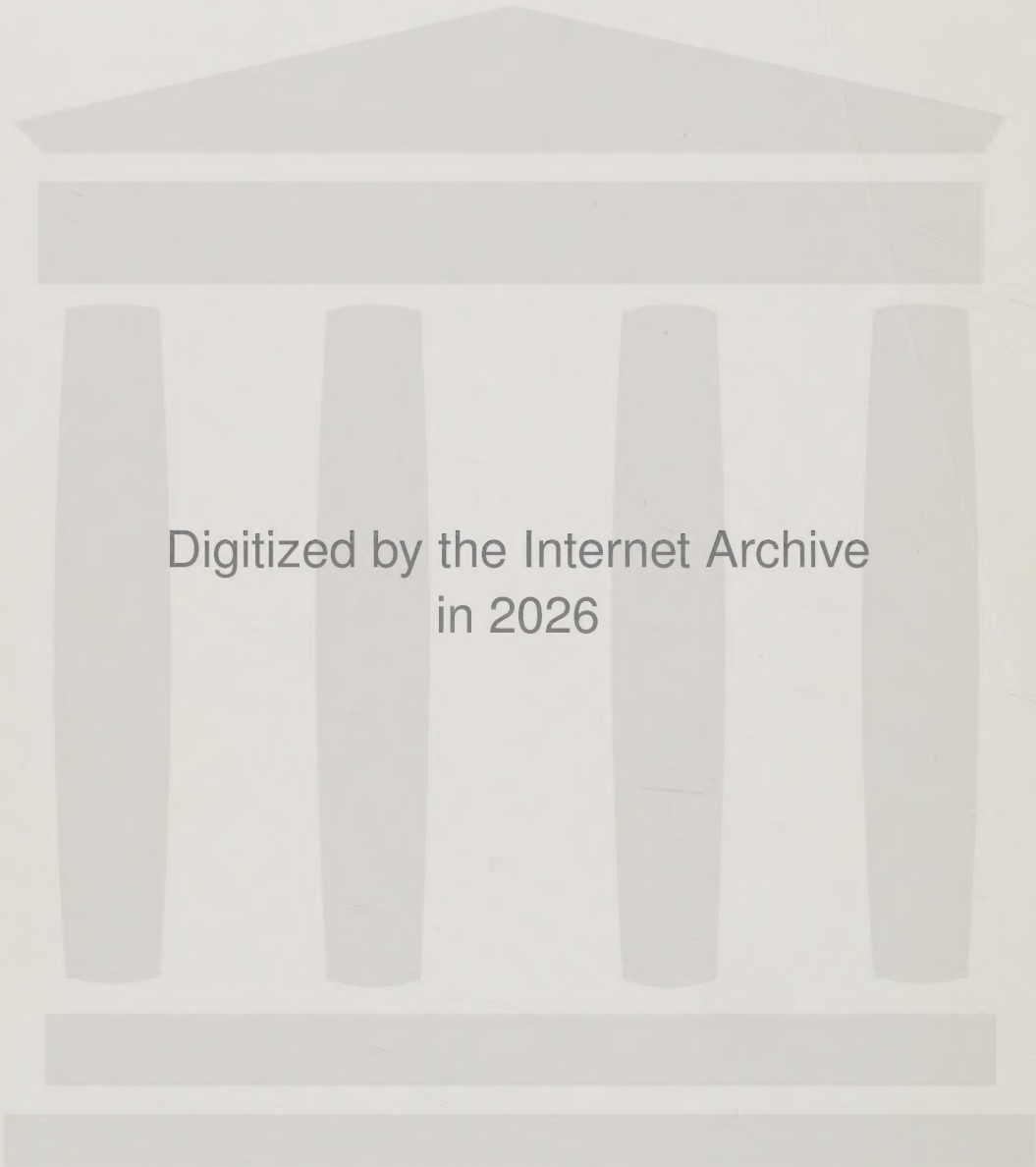



# SELF-CORRECTING PROBLEMS IN STATISTICS

G. A. Whitmore
John Neter
William Wasserman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# Self-Correcting Problems in STATISTICS

# Self-Correcting Problems in STATISTICS

*G. A. Whitmore*
*McGill University*

*John Neter*
*University of Minnesota*

*William Wasserman*
*Syracuse University*

*Allyn and Bacon, Inc., Boston*

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# Contents


| | | |
|---|---|---|
| | Preface | vii |
| | Index of Symbols | x |
| **1.** | Ratios | 1 |
| **2.** | Frequency Distributions | 9 |
| **3.** | Characteristics of Frequency Distributions: Averages | 17 |
| **4.** | Characteristics of Frequency Distributions: Measures of Dispersion and Skewness | 27 |
| **5.** | Introduction to Probability | 35 |
| **6.** | Random Variables | 49 |
| **7.** | Common Probability Distributions | 59 |
| **8.** | Introduction to Sampling Distributions | 71 |
| **9.** | Sampling Distribution of $\overline{X}$ | 79 |
| **10.** | Sampling Distribution of $\overline{p}$ | 85 |
| **11.** | Interval Estimation of the Population Mean — Large Simple Random Sample | 93 |
| **12.** | Interval Estimation of the Difference Between Two Population Means — Large Simple Random Samples | 99 |
| **13.** | Interval Estimation of the Population Mean — Normal Population, Small Sample | 103 |
| **14.** | Interval Estimation of the Population Proportion — Large Simple Random Sample | 109 |
| **15.** | Introduction to Decision-Making | 115 |
| **16.** | Statistical Decision-Making: Hypothesis Testing — Population Mean, Large Sample | 123 |
| **17.** | Construction of Decision Rules Concerning Population Mean — Large Sample | 135 |
| **18.** | Construction of Decision Rules Concerning the Difference Between Two Population Means — Large Samples | 147 |
| **19.** | Construction of Decision Rules Concerning Population Mean — Normal Population, Small Sample | 153 |

| | | |
|---|---|---|
| **20.** | Construction of Decision Rules Concerning Population Proportion — Large Sample | 159 |
| **21.** | Tests of Goodness of Fit (Chi-Square Tests) | 171 |
| **22.** | Contingency Tables (Chi-Square Tests) | 183 |
| **23.** | One-Way Analysis of Variance | 193 |
| **24.** | Two-Way Analysis of Variance | 203 |
| **25.** | Inferences Concerning Variances | 213 |
| **26.** | Simple Regression Analysis | 219 |
| **27.** | Multiple and Curvilinear Regression Analysis | 231 |
| **28.** | Correlation Analysis | 239 |
| **29.** | Statistical Decision-Making Using Payoffs | 247 |
| **30.** | Bayesian Decision-Making | 257 |
| **31.** | Normal Prior Distribution and Normal Sampling | 265 |
| **32.** | Index Numbers | 277 |
| **33.** | Trends | 287 |
| **34.** | Seasonal Variation | 295 |
| **35.** | Cyclical Fluctuations | 305 |
| **36.** | Additional Forecasting Techniques | 311 |
| | APPENDIX (Table A-1–Table A-11) | 317 |
| | Reference Table to Standard Statistics Textbooks | 339 |

# Preface

**How to Use Self-Correcting Problems Workbook**

This workbook is designed for studying statistical methods by solving "self-correcting" problems. It contains 36 sections, each covering a series of topics. Presented in each topic is an explanation of the subject taken up, an expository example, and one or more problems to be solved. In solving these problems, you check your answers against those found in the workbook. The problems play a central role in the self-teaching orientation of the workbook—you will be learning from the questions that you answer. A correct answer shows that you are on the right track and will reinforce your grasp of the material. Should your answer be wrong, you will be able to locate the mistake at once, and discover for yourself why you made it – for example, you may find that you do not yet fully understand some point or concept. To assist you, the answers in this workbook are fully developed, with calculations systematically set out and bases for verbal answers explained.

In terms of self-teaching, it is important that you do not proceed to the next block of material until you have correctly answered the questions at hand. If you make a mistake, identify the trouble and redo the questions. Then check your answers again and go forward or repeat as the case may be. You will find this approach both time-saving and effective.

A checkmark ($\surd$) is employed to indicate the points where you should compare your answers with those in the workbook. Problems that cover a number of points are broken into smaller steps, and frequent checkpoints are used so that you cannot go too far in the wrong direction before identifying and correcting your errors. There are also review problems at the end of each section; here more ground is covered between checkpoints. Unless otherwise indicated, answers will be found on the first even-numbered page following the checkpoint. Thus, for a checkpoint on page 1, the answer is found on page 2; for checkpoints on pages 2 and 3, the answers are found on page 4; and so on. Exceptions to this practice are noted in the text, and the page number where the answer is located is given at the checkpoint. Answers are boxed and shaded for easy identification.

Most sections can be completed in one sitting. However, a few sections may require more time to work out. These include Sections 5, 20, 21, 23, 24, 26, 27, 31, and 32.

Except in selected cases, blank space is provided in the workbook only for final calculations or short answers. We urge that you systematically write supporting calculations or longer verbal answers on sheets of three-ring notebook paper and retain these, as later problems may utilize calculations made earlier. These notebook pages and detached workbook pages can be collated and filed in a three-ring binder for easy reference and review.

Since a variety of notation is used in different statistics textbooks, the notation utilized in this workbook may not always agree with that of your text. To help you with the notation used in this workbook, an index of symbols is included immediately before Section 1 so that you can easily find the place where each symbol is defined.

Tables and Figures in each section are identified by the section number followed by a table or figure number. Tables in the Appendix at the back of the workbook are designated Table A-1, etc.

Also at the back is a reference table that indicates for a large number of statistics texts the sections in this workbook that should be completed for each textbook chapter. You will find it is generally desirable to study the textbook chapter first and then work through the corresponding sections of this workbook. Sections in parentheses in the reference table cover materials not treated directly in the textbook, but which build on concepts discussed in the textbook chapter.

At times you may find that your answer differs slightly from that given in this workbook, even though the procedure you followed is correct. Such a difference is likely due to rounding and need not concern you, provided that your procedure is correct.

### To Teachers of Statistics

This self-correcting problems workbook may be used in undergraduate and graduate introductory or intermediate courses in business statistics, economic statistics, or general applied statistics, as well as in some management science courses. The scope of the workbook includes descriptive and inferential statistics, classical and Bayesian statistics, and introductory as well as intermediate topics such as analysis of variance and multiple regression. The topics are systematically cross-referenced so that they can be taken up in a different order than used here, and some topics may be omitted completely if desired. The reference table at the end of the workbook shows which workbook sections pair with each textbook chapter for a selection of standard statistics textbooks in business, economic, and general statistics.

This workbook differs from an ordinary laboratory manual by containing self-correcting problems and expository material. The self-correcting problems emphasize understanding, and calculations are kept to a minimum. The self-correcting feature of the problems incorporates principles of programmed learning, in that correct understanding is immediately reinforced. At the same time, causes of errors are identified so that correct understanding can be achieved. Frequent checkpoints in the problem material enable the student to be rewarded often and guided against going too far in the wrong direction. Use of this workbook in conjunction with a textbook and class instruction (or independent study for review or refreshing) can make the study of statistics much more effective.

The problem material is realistic, designed to motivate the student and to indicate the usefulness of statistical methods. Not only are there self-correcting problems for each topic, but each of the 36 sections also contains a set of comprehensive review problems at the end.

### Acknowledgments

We should like to acknowledge the expeditious typing of the manuscript by Mrs. Eileen Bonnor, the extensive checking for errors by Roy Koenig, and the suggestions on the manuscript by Robert W. Johnson and Roland I. Robinson.

We are also indebted to the Literary Executor of the late Sir Ronald A. Fisher, F.R.S., to Dr. Frank Yates, F.R.S., and to Oliver and Boyd Ltd., Edinburgh, for permission to reprint Tables III and IV from their book *Statistical Tables for Biological, Agricultural and Medical Research.*

Since it is inevitable that some errors are present in the workbook, despite careful checking of manuscript and proof, we would appreciate having these errors called to our attention.

*G.A. Whitmore*

*John Neter*

*William Wasserman*

# Index of Symbols

The following is a list of standard and special symbols and notation used in this book, with a brief description of each and the page where it first appears.

| Symbol | Description | Page |
|---|---|---|
| **LATIN SYMBOLS** | | |
| $a$ | number of factor $A$ classes | 204 |
| | estimator of intercept $\alpha$ | 219 |
| $A$ | action limit | 135 |
| | left tail $\chi^2$ value | 213 |
| $A_1, A_2$ | action limits | 135 |
| $b$ | number of factor $B$ classes | 204 |
| | estimator of slope $\beta$ | 219 |
| $B$ | right tail $\chi^2$ value | 213 |
| $c$ | cost of sample observation | 261 |
| $C$ | number of column classes | 184 |
| | left tail $F$ value | 215 |
| $C_1, C_2$ | alternative acts | 123 |
| $\mathrm{Cov}(X,Y)$ | covariance of $x$ and $y$ | 52 |
| $d$ | difference of sample proportions | 111 |
| $df$ | degrees of freedom | 197 |
| $D$ | difference of sample means | 99 |
| | right tail $F$ value | 215 |
| | unit normal deviate | 267 |
| $D*$ | $D$ value for sample information | 269 |
| $e$ | constant, about 2.718 | 63 |
| $ENGS$ | expected net gain from sampling | 271 |
| $EVPI$ | expected value – perfect information | 261 |
| $EVSI$ | expected value – sample information | 269 |
| $E(x)$ | expected value of $x$ | 49 |
| $f$ | class frequency | 21 |
| $f_e$ | expected frequency | 172 |
| $f_s$ | sample frequency | 172 |
| $F$ | $F$ distribution value | 193 |
| $h$ | half-width | 94 |
| $i$ | class width | 21 |
| $k$ | number of standard deviations | 55 |
| | number of classes | 173 |
| $K$ | indifference point | 266 |
| $L$ | lower limit of median class | 21 |
| $L(D)$ | value, unit normal loss function | 267 |
| $m$ | parameter of $t$ distribution | 103 |
| | number of estimated parameters | 175 |
| | number of populations | 187 |
| $M$ | mean of $\mu$ | 267 |
| $M_0$ | prior mean of $\mu$ | 265 |
| $M_1$ | posterior mean of $\mu$ | 267 |
| $MS, MS_1, MS_2$ | one-way mean squares | 197 |
| $MS_A, MS_B, MS_{AB}$ | two-way mean squares | 206 |
| $MS_E$ | residual mean square: simple regression | 225 |
| | multiple regression | 235 |
| $MS_R$ | simple regression mean square | 225 |
| | multiple regression mean square | 235 |
| $n$ | number of items (sample size) | 17 |
| | sum of class frequencies | 21 |
| $n_1, n_2$ | frequencies for computing median | 21 |
| $N$ | finite population size | 32 |
| $OL_1, OL_2$ | opportunity loss functions | 266 |
| $p$ | binomial parameter | 59 |
| | population proportion | 85 |
| $\bar{p}$ | sample proportion | 61 |
| $p_0, p_n$ | unit prices | 277 |
| $p_t$ | typical unit price | 279 |
| $P_n$ | price index of $n$th period | 277 |
| $P(X)$ | probability of $X$ | 37 |
| $P(Y\|X)$ | probability of $Y$ given $X$ | 38 |
| $P_1, P_2$ | payoff curves | 265 |
| $q_0, q_n$ | quantities | 279 |
| $q_t$ | typical quantity | 277 |
| $Q$ | total $SS$: analysis of variance | 196 |
| | regression | 225 |
| $Q_n$ | quantity index of $n$th period | 279 |
| $Q_1, Q_2$ | one-way sums of squares | 196 |
| $Q_A, Q_B, Q_{AB}$ | two-way sums of squares | 204 |
| $Q_E$ | residual $SS$: simple regression | 225 |
| | multiple regression | 234 |
| $Q_R$ | simple regression $SS$ | 225 |
| | multiple regression $SS$ | 234 |
| $r$ | sample coefficient of correlation | 239 |
| | rank correlation coefficient | 244 |
| $r^2$ | sample coefficient of determination | 225 |
| $r_{ij.k}$ | sample partial correlation coefficient | 243 |
| $R$ | number of row classes | 184 |
| | sample multiple correlation coefficient | 242 |
| $R^2$ | sample coefficient of multiple determination | 235 |

| Symbol | Description | Page |
|---|---|---|
| $s$ | sample standard deviation: | |
| | grouped data | 29 |
| | ungrouped data | 28 |
| $s^2$ | sample variance | 28 |
| | pooled sample variance | 105 |
| $s_b$ | estimator of standard deviation: of $b$ | 223 |
| $s_d$ | of $d$ | 111 |
| $s_D$ | of $D$ | 99 |
| $s_{\bar{p}}$ | of $\bar{p}$ | 109 |
| $s_{\bar{X}}$ | of $\bar{X}$ | 93 |
| $s_{Y.X}$ | estimator of $\sigma_{Y.X}$ | 219 |
| $s_{Y.X_1X_2}$ | estimator of $\sigma_{Y.X_1X_2}$ | 232 |
| $S$ | standard deviation of $\mu$ | 267 |
| $S_0$ | prior standard deviation of $\mu$ | 265 |
| $S_1$ | posterior standard deviation of $\mu$ | 267 |
| $S*$ | $S$ value for sample information | 269 |
| $SS$ | sum of squares | 197 |
| $t$ | $t$ distribution value | 103 |
| | number of treatments | 197 |
| | slope of loss function | 267 |
| $T$ | estimator of population total | 72 |
| $V$ | test statistic: for $\rho = 0$ | 240 |
| $V_1$ | $\chi^2$ tests | 172 |
| $V_2$ | one-way | 198 |
| $V_3$ | for $\sigma^2 = \sigma_0^2$ | 214 |
| $V_4$ | for $\sigma_1^2 = \sigma_2^2$ | 216 |
| $V_A$, $V_B$, $V_{AB}$ | two-way | 207 |
| $V_R$ | simple regression | 225 |
| | multiple regression | 235 |
| $w$ | Fisher's transformation variable | 241 |
| | exponential smoothing weight | 312 |
| $X^*$ | specified $X$ value | 221 |
| $\bar{X}$ | sample mean: | |
| | grouped data | 21 |
| | ungrouped data | 17 |
| $\bar{X}_{i.}$ | factor or treatment sample mean | 204 |
| $\bar{X}_{.j}$ | factor or treatment sample mean | 196 |
| $\bar{X}_{..}$ | sample grand mean | 196 |
| $Y^*$ | new $Y$ observation | 222 |
| $\bar{Y}_X$ | estimator of $\mu_{Y.X}$ | 220 |
| $\bar{Y}_{X_1X_2}$ | estimator of $\mu_{Y.X_1X_2}$ | 233 |
| $z$ | standard normal deviate | 63 |

**GREEK SYMBOLS**

| Symbol | Description | Page |
|---|---|---|
| $\alpha$ | error probability, interval estimate | 213 |
| | regression intercept | 219 |
| $\alpha_i$ | factor $A$ effect | 205 |
| $\alpha_j$ | treatment effect | 195 |
| $\beta$ | regression slope, simple | 219 |
| $\beta_1, \beta_2$ | regression parameters, multiple | 231 |
| $\beta_j$ | factor $B$ effect | 205 |
| $\gamma_{ij}$ | interaction effect | 205 |
| $\delta$ | parameter of $\chi^2$ distribution | 171 |
| $\delta_1, \delta_2$ | parameters of $F$ distribution | 193 |
| $\epsilon$ | residual error component | 195 |
| $\mu$ | population mean | 32 |
| | mean of probability distribution | 50 |
| | overall true mean | 195 |
| $\mu_0$ | specified $\mu$ value | 135 |
| $\mu_d$ | mean: of $d$ | 163 |
| $\mu_D$ | of $D$ | 147 |
| $\mu_{\bar{p}}$ | of $\bar{p}$ (finite population) | 86 |
| | of $\bar{p}$ (infinite population) | 87 |
| $\mu_{\bar{X}}$ | of $\bar{X}$ (finite population) | 80 |
| | of $\bar{X}$ (infinite population) | 80 |
| $\mu_{Y.X}$ | of conditional distribution of $Y$ | 219 |
| $\pi$ | constant, about 3.142 | 63 |
| $\rho$ | correlation coefficient | 239 |
| $\rho^2$ | coefficient of determination | 239 |
| $\rho_{ij.k}$ | partial correlation coefficient | 243 |
| $\rho_m$ | multiple correlation coefficient | 242 |
| $\rho_m^2$ | coefficient of multiple determination | 242 |
| $\sigma$ | population standard deviation | 32 |
| | standard deviation of probability distribution | 50 |
| $\sigma^2$ | variance of probability distribution | 50 |
| $\sigma_0$ | specified $\sigma$ value | 214 |
| $\sigma_d$ | standard deviation: of $d$ | 111 |
| $\sigma_D$ | of $D$ | 99 |
| $\sigma_{\bar{p}}$ | of $\bar{p}$ (finite population) | 86 |
| | of $\bar{p}$ (infinite population) | 87 |
| $\sigma_{\bar{X}}$ | of $\bar{X}$ (finite population) | 80 |
| | of $\bar{X}$ (infinite population) | 80 |
| $\sigma_{Y.X}$ | of conditional distribution of $Y$ | 219 |
| $\Sigma$ | single summation operation | 17 |
| $\Sigma\Sigma$ | double summation operation | 193 |
| $\tau$ | population total | 72 |
| $\chi^2$ | chi-square distribution value | 171 |

**OTHER SYMBOLS**

| Symbol | Description | Page |
|---|---|---|
| $\binom{n}{X}$ | binomial coefficient | 59 |
| $\lvert\ \rvert$ | absolute value | 27 |
| ! | factorial | 59 |
| $\leqslant$ | less than or equal to | |
| $\geqslant$ | greater than or equal to | |

# 1

# Ratios

Ratios express relationships between quantitative variables and thereby aid in summarizing and interpreting statistical data.

**A. Construction of Statistical Ratios.** A statistical ratio expresses the magnitude of the variable in the numerator in relation to a given number of units of the variable in the denominator.

Example 1 In 1968, the P & A Railroad Co. had freight revenues of $6,313,300 and freight loss and damage expenses of $26,800. Hence, the number of dollars of freight loss and damage expenses per thousand dollars of freight revenue was 4.24:

$$\frac{26{,}800}{6{,}313{,}300} \times 1000 = 4.24.$$

The formal proportion involved is:

$$26{,}800 : 6{,}313{,}300 = \text{ratio} : 1000.$$

From this proportion, we obtain:

$$\frac{26{,}800}{6{,}313{,}300} = \frac{\text{ratio}}{1000} \quad \text{which gives} \quad \text{ratio} = \frac{26{,}800}{6{,}313{,}300} \times 1000 = 4.24.$$

**Problem 1**

(a) In 1968, the Northern Maine Electric Company had sales revenue of $143 million from the sale of 5700 million kilowatt hours of electric power. Calculate the sales revenue per thousand kilowatt hours. _____ √

(b) In 1965, disposable personal income in the United States was $468 billion and the population was 195 million people. Calculate the disposable personal income per capita (per person) in 1965. _____ √

(c) In 1967, a bank's common stock sold at $11.20 per share. In that year it had per share earnings of $.70 and paid a $.49 dividend.

(i) Calculate the bank's price/earnings ratio (price per dollar of earnings). _____

(ii) Calculate the per cent of earnings paid out in dividends. _____ √

**B. Per Cent Distributions.** A per cent distribution shows each component of a total as a per cent of the total.

Example 2 Analysis of financial data often requires conversion of absolute data into per cents. Given in Table 1.1 are data on sales and costs of the Hill Equipment Company for 1967 and 1968. The per cent distribution of sales for 1967 is shown. It is obtained by expressing each accounting entry as a per cent of 1967 sales.

**Answers**

1. (a) $\frac{143,000,000}{5,700,000,000} \times 1000 = \$25$ per thousand kilowatt hours.

   (b) $\frac{468,000,000,000}{195,000,000} \times 1 = \$2400$ per capita.

   (c) (i) Price/earnings ratio = 11.20/.70 = 16.0;
   (ii) Dividend payout = (.49/.70) X 100 = 70%.

Table 1.1

| | 1967 | Per Cent of Sales | 1968 | Per Cent of Sales |
|---|---|---|---|---|
| Sales | $200,000 | 100.0 | $250,000 | _____ |
| Cost of goods sold | 130,000 | 65.0 | 170,000 | _____ |
| Gross margin | $ 70,000 | 35.0 | $ 80,000 | _____ |
| Expenses | 65,000 | 32.5 | 75,000 | _____ |
| Net profit before income taxes | $ 5,000 | 2.5 | $ 5,000 | _____ |

**Problem 2**

(a) Determine the per cent distribution of sales for 1968 in Example 2.
(b) Does lower profit as a per cent of sales in 1968 imply lower total profit? _____√

**C. Use of Ratios to Study Relative Changes From Base Periods.** Ratios can be used to study relative changes over time.

Example 3 If one is interested in relative growth or decline from a base period, one can relate the magnitude of a variable in a given period to the base period magnitude. This ratio, when expressed as a per cent, is called a *per cent relative.*

Sales revenues for two milling companies from 1965 to 1968 inclusive are shown in Table 1.2.

Table 1.2

| Year | Sales Revenue for Company A ($ millions) | Per Cent Relative | Sales Revenue for Company B ($ millions) | Per Cent Relative |
|---|---|---|---|---|
| 1965 | 525 | 100 | 420 | _____ |
| 1966 | 540 | 103 | 445 | _____ |
| 1967 | 560 | 107 | 445 | _____ |
| 1968 | 525 | 100 | 470 | _____ |

The per cent relatives for the sales revenue data are calculated as follows:

$$\text{per cent relative for given period} = \frac{\text{sales revenue in given period}}{\text{sales revenue in base period}} \times 100.$$

For Company A, if 1965 is to be used as the base period, the per cent relative for 1966 is obtained as follows:

$$\text{per cent relative for 1966} = \frac{540}{525} \times 100 = 103.$$

Note that the base period per cent relative equals 100 by definition.

**Problem 3**

Using 1965 as the base period, calculate per cent relatives for the sales revenue data of Company B in Table 1.2. √

Example 4 Sometimes it is desirable to analyze a variable by making comparisons between adjacent time periods, instead of between each period and a fixed base period. Such comparisons are facilitated by calculating *link relatives.* A link relative expresses the magnitude of the variable in a given period as a per cent of the magnitude of the variable in the immediately preceding period.

Consider the sales revenue data of the two milling companies given in Example 3.

**Table 1.3**

| Year | Sales Revenue of Company A | Link Relative | Sales Revenue of Company B | Link Relative |
|---|---|---|---|---|
| 1965 | 525 | – | 420 | _____ |
| 1966 | 540 | 103 | 445 | _____ |
| 1967 | 560 | 104 | 445 | _____ |
| 1968 | 525 | 94 | 470 | _____ |

The link relatives for Company A are shown in Table 1.3. The link relatives are calculated using the following relation:

$$\text{link relative for given period} = \frac{\text{sales revenue in given period}}{\text{sales revenue in preceding period}} \times 100.$$

For instance, for Company A, the link relative for 1968 is obtained as follows:

$$\text{link relative for 1968} = \frac{525}{560} \times 100 = 94.$$

**Problem 4**

(a) Why is no link relative shown for Company A in 1965? _____

(b) Calculate the link relatives for Company B. √

Example 5 Another method for comparing magnitudes of a variable in two periods is to calculate *per cent changes.* In general, we calculate a per cent change as follows:

$$\text{per cent change} = \left(\frac{\text{magnitude in later period}}{\text{magnitude in earlier period}} \times 100\right) - 100.$$

Consider the sales revenue data for Company A in Table 1.3. The per cent change in sales revenue between 1965 and 1967 is:

$$\left(\frac{560}{525} \times 100\right) - 100 = 7\%.$$

Hence sales revenue increased 7% between 1965 and 1967.

**Answers**

2. (a) Per Cent Distribution

| |
|---|
| 100.0 |
| 68.0 |
| 32.0 |
| 30.0 |
| 2.0 |

(b) No, net profit before income taxes was $5000 in both years.

3.

| Year | Per Cent Relative |
|---|---|
| 1965 | 100 |
| 1966 | 106 |
| 1967 | 106 |
| 1968 | 112 |

4. (a) No link relative can be shown for 1965 because the sales revenue in 1964 is not given.

(b)

| Year | Link Relative |
|---|---|
| 1965 | – |
| 1966 | 106 |
| 1967 | 100 |
| 1968 | 106 |

**Problem 5**

(a) For Company A, calculate the per cent change in sales revenue between 1966 and 1968 (data in Example 4).

(b) A government bureau had a budget of $28 million in 1967 and a budget of $32 million in 1968. By what per cent did this bureau's budget increase between 1967 and 1968? √

Example 6 A common problem with per cent data is the distinction between *per cent points of change* and *per cent changes.*

In 1965 the unemployment rate in the labor force was 4.6%. In 1966, it was 4.0%. Between 1965 and 1966 there was a .6 per cent point decrease in the unemployment rate. This represents a 13% decrease:

$$\left(\frac{4.0}{4.6} \times 100\right) - 100 = -13\%.$$

**Problem 6**

In 1964, 10.8% of the federal budget was used to pay interest on public debt. In 1965, the percentage was 11.7.

(a) Calculate the per cent change between 1964 and 1965 in the percentage of the federal budget used to pay interest on public debt.

(b) By how many per cent points did the percentage of the federal budget used to pay interest on public debt change between 1964 and 1965? √

**D. Averaging of Ratios.** When ratios are averaged, relevant "weights" usually must be employed to obtain meaningful results.

Example 7 A real estate firm manages three apartment blocks, A, B, and C. The vacancy rate and number of units in each apartment block is shown in Table 1.4. To obtain the vacancy rate for the group of three apartment blocks, we employ a weighted average with the number of units as weights. The vacancy rates of the individual apartment blocks are multiplied by their respective number of units, these products are totaled, and this total is divided by the sum of the weights. The resulting value is the weighted average vacancy rate.

**Table 1.4**

| Apartment Block | (Ratio) Vacancy Rate | (Weight) Number of Units | (Ratio X Weight) Vacancy Rate X Number of Units |
|---|---|---|---|
| A | 4.5 | 80 | 360 |
| B | 2.0 | 20 | 40 |
| C | 7.0 | 50 | 350 |
| Total | | 150 | 750 |

$$\text{weighted average vacancy rate} = \frac{750}{150} = 5.0.$$

**Problem 7**

The Superior Company has three plants. During July, each operated at the following per cent of capacity:

| Plant | Per Cent of Capacity |
|---|---|
| A | 72 |
| B | 80 |
| C | 61 |

Additional information pertaining to July is shown in Table 1.5.

**Table 1.5**

| Plant | Number of Employees | Monthly Capacity (no. of production units) | Ranking as to Efficiency |
|---|---|---|---|
| A | 2000 | 50,000 | 2 |
| B | 3000 | 80,000 | 3 |
| C | 4950 | 100,000 | 1 |

(a) Which series in Table 1.5 represents the relevant weights for obtaining the overall per cent of capacity for the three plants? _____ √

(b) Calculate the per cent of capacity utilized in July for the entire company. _____ √

Answers

5. (a) $\left(\frac{525}{540} \times 100\right) - 100 = -3$. Sales revenue declined 3% between 1966 and 1968.

   (b) $\left(\frac{32}{28} \times 100\right) - 100 = 14$. The bureau's budget increased 14% between 1967 and 1968.

6. (a) Per cent change $= \left(\frac{11.7}{10.8} \times 100\right) - 100 = 8\%$.

   (b) Per cent points of change $= 11.7 - 10.8 = .9$.

7. (a) Monthly capacity, because it is the denominator of the ratio, per cent of capacity.

(b)

| Plant | (Ratio) Per Cent of Capacity | (Weight) Monthly Capacity | (Ratio X Weight) |
|---|---|---|---|
| A | 72 | 50,000 | 3,600,000 |
| B | 80 | 80,000 | 6,400,000 |
| C | 61 | 100,000 | 6,100,000 |
| Total | | 230,000 | 16,100,000 |

Per cent of capacity utilized for entire company $= \frac{16{,}100{,}000}{230{,}000} = 70$.

## REVIEW PROBLEMS

1. In the *Failure Record Through 1966*, published by Dun & Bradstreet, Inc., it is reported that of the total number of business enterprises listed in the *Dun & Bradstreet Reference Book*, about 13,000 businesses failed in 1966 leaving total liabilities of $1391 million.
   (a) What is the average liability per business failure? _____
   (b) It was also reported by Dun & Bradstreet that the failure rate per 10,000 businesses listed in their Reference Book was 52. Determine the approximate number of business enterprises listed in the *Dun & Bradstreet Reference Book*. _____ √

2. Table 1.6 shows the number of persons employed in various occupation groups in 1965. Construct a per cent distribution for these data.

Table 1.6

| Occupation Group | Number of Employed Persons (in millions) | Per Cent Distribution |
|---|---|---|
| White-collar workers | 32 | _____ |
| Blue-collar workers | 27 | _____ |
| Service workers | 9 | _____ |
| Farm workers | 4 | _____ |
| Total | 72 | _____ √ |

3. In April, 1968, the P & A Railroad reported "first quarter earnings of 58 cents a share, up 16% from one year earlier." What were the first quarter per share earnings in 1967? _____ √

4. The per cent relatives for the amount of insurance in force of a midwest life insurance company, for the period 1965 to 1968 inclusive, are given in Table 1.7.

**Table 1.7**

| Year | Per Cent Relative, Amount of Insurance in Force | Link Relative, Amount of Insurance in Force |
|---|---|---|
| 1965 | 80 | _____ |
| 1966 | 86 | _____ |
| 1967 | 92 | _____ |
| 1968 | 100 | _____ |

(a) What year has been used as the base period of the per cent relatives? _____

(b) Calculate the link relatives for the amount of insurance in force from the per cent relatives for the same data. √

5. On April 22, a treasury bond sold at a yield of 4.40%. One year later, this same treasury bond sold at a yield of 5.50%.

(a) What per cent increase in the bond yield occurred during the year? _____

(b) By how many per cent points did the bond yield change during the year? _____ √

6. A firm has three capital investment projects. The investment outlays and annual rates of return for the projects are shown in Table 1.8. Calculate the rate of return on the total investment in the three projects. _____ √

**Table 1.8**

| Project | Per Cent Rate of Return | Investment Outlay (in thousands) |
|---|---|---|
| A | 16 | 100 |
| B | 20 | 60 |
| C | 10 | 40 |

## ANSWERS TO REVIEW PROBLEMS

1. (a) $\frac{1{,}391{,}000{,}000}{13{,}000} \times 1 = \$107{,}000$ per business failure.

   (b) $\frac{13{,}000}{52} \times 10{,}000 = 2{,}500{,}000$ business enterprises.

2.

| Occupation Group | Per Cent Distribution |
|---|---|
| White-collar workers | 44 |
| Blue-collar workers | 38 |
| Service workers | 12 |
| Farm workers | 6 |
| Total | 100 |

3. First quarter per share earnings $= 58 \times \frac{100}{116} = 50$ cents.

4. (a) 1968

   (b)

| Year | Link Relative |
|---|---|
| 1965 | — |
| 1966 | 108 |
| 1967 | 107 |
| 1968 | 109 |

5. (a) $\left(\frac{5.5}{4.4} \times 100\right) - 100 = 25\%$ increase.

   (b) $5.5 - 4.4 = 1.1$ per cent points of change.

6.

| Project | (Ratio) Rate of Return | (Weight) Investment | (Ratio X Weight) |
|---|---|---|---|
| A | 16 | 100 | 1600 |
| B | 20 | 60 | 1200 |
| C | 10 | 40 | 400 |
| Total | | 200 | 3200 |

Weighted average rate of return $= \frac{3200}{200} = 16\%$.

# 2

# Frequency Distributions

A frequency distribution is the classification of a group of items by some quantitative characteristic. The number of items in a class is called the *frequency* of that class.

**A. Construction of Frequency Distributions.**

Example 1 The following are the amounts of freight charges due Crandell Freight, Inc. on 35 freight bills issued on November 1, 1968.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2.03 | 1.82 | .88 | 1.62 | 5.89 | 2.02 | 2.63 | 1.80 | 3.02 |
| 2.57 | 2.81 | 1.86 | 4.81 | 1.10 | 4.04 | 2.00 | 1.89 | 2.30 |
| 1.59 | 1.77 | 1.70 | .86 | 1.87 | 3.62 | .99 | 2.65 | 1.56 |
| 2.57 | 3.02 | 2.91 | 2.07 | 2.15 | 1.16 | 3.24 | 3.44 | |

Table 2.1 presents a form that is convenient to use when tallying such ungrouped data to see how many items fall in each class; it also gives a rough picture of the pattern of variation once the tallying has been completed. The frequency of each class is shown in the table. By expressing each frequency as a per cent of the total number of freight bills, a *per cent frequency distribution* is obtained.

**Table 2.1**

| Amount of Freight Charges | Tally | Number of Freight Bills | Per Cent of Freight Bills |
|---|---|---|---|
| 0.00–0.99 | /// | 3 | 9 |
| 1.00–1.99 | ~~HH~~ ~~HH~~ // | 12 | 34 |
| 2.00–2.99 | ~~HH~~ ~~HH~~ // | 12 | 34 |
| 3.00–3.99 | ~~HH~~ | 5 | 14 |
| 4.00–4.99 | // | 2 | 6 |
| 5.00–5.99 | / | 1 | 3 |
| Total | | 35 | 100 |

**Problem 1**

The following is the net return on investment in 1968 for each of 60 life insurance companies. Construct a frequency distribution for these 60 values using the indicated classes. Also obtain the per cent frequency distribution, using Table 2.2 on page 10 to fill in your answers.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5.85 | 6.18 | 6.30 | 6.08 | 5.26 | 5.30 | 4.18 | 4.20 | 5.71 | 5.92 |
| 6.04 | 5.81 | 6.14 | 6.02 | 5.87 | 3.96 | 5.92 | 5.66 | 5.27 | 4.34 |
| 3.75 | 6.39 | 5.04 | 4.06 | 6.12 | 5.96 | 4.55 | 4.72 | 4.89 | 5.62 |
| 6.14 | 5.67 | 5.01 | 5.58 | 6.18 | 5.24 | 4.98 | 6.01 | 4.92 | 6.18 |
| 4.94 | 5.96 | 5.70 | 5.98 | 4.84 | 5.74 | 6.02 | 4.46 | 6.13 | 4.30 |
| 4.98 | 6.54 | 6.37 | 5.85 | 5.96 | 5.04 | 5.28 | 5.08 | 5.84 | 4.64 |

Table 2.2

| Net Return on Investment | Number of Companies | Per Cent of Companies |
|---|---|---|
| 3.50–3.99 | ——— | ——— |
| 4.00–4.49 | ——— | ——— |
| 4.50–4.99 | ——— | ——— |
| 5.00–5.49 | ——— | ——— |
| 5.50–5.99 | ——— | ——— |
| 6.00–6.49 | ——— | ——— |
| 6.50–6.99 | ——— | ——— |
| Total | ═══ | ═══ √ |

**B. Construction of Cumulative Frequency Distributions.** In some situations, it is desired to have information about the number or the per cent of items "less than" or "more than" any given value. The resulting distribution is called a "less than" cumulative frequency distribution or a "more than" cumulative frequency distribution. A cumulative frequency distribution can be constructed in per cent form or in terms of the actual number of observations.

Example 2 Consider the frequency distribution of freight charges developed in Example 1. Shown in Table 2.3 are the "less than" cumulative frequency distributions, both on absolute and per cent bases, for the data.

**Table 2.3**

| Amount of Freight Charges | Cumulative Number of Freight Bills | Cumulative Per Cent of Freight Bills |
|---|---|---|
| Less than 0 | 0 | 0 |
| Less than 1.00 | 3 | 9 |
| Less than 2.00 | 15 | 43 |
| Less than 3.00 | 27 | 77 |
| Less than 4.00 | 32 | 91 |
| Less than 5.00 | 34 | 97 |
| Less than 6.00 | 35 | 100 |

**Problem 2**

Consider the frequency distribution developed in Example 1. Complete the "more than" cumulative frequency distributions, both on absolute and per cent bases, for the Crandell data:

**Table 2.4**

| Amount of Freight Charges | Cumulative Number of Freight Bills | Cumulative Per Cent of Freight Bills |
|---|---|---|
| 0.00 or more | 35 | 100 |
| 1.00 or more | 32 | —— |
| ——— | — | — |
| ——— | — | — |
| ——— | — | — |
| ——— | — | — |
| ——— | — | — √ |

### C. Graphic Presentation — Frequency Polygons, Histograms, Ogives.

Example 3 To construct a *frequency polygon*, each class frequency is plotted corresponding to the midpoint of its class. The plotted points are then connected by a series of straight lines. Shown in Figure 2.1 is the frequency polygon of the frequency distribution developed in Example 1.

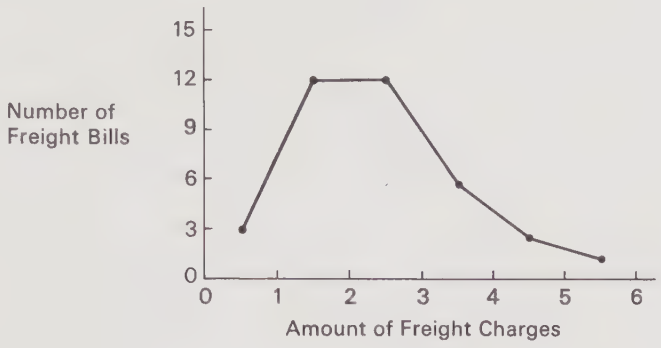
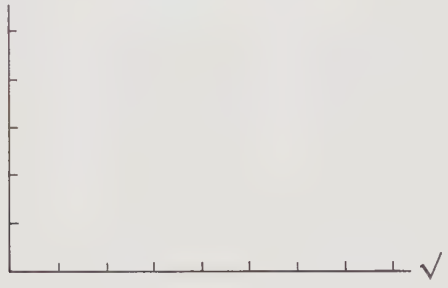



**Figure 2.1**

**Problem 3**

Construct the frequency polygon for the frequency distribution in Problem 1.



Example 4 To construct a *histogram*, bars are drawn that span each class interval, and have heights that correspond to the class frequencies. Shown in Figure 2.2 is the histogram of the frequency distribution developed in Example 1.

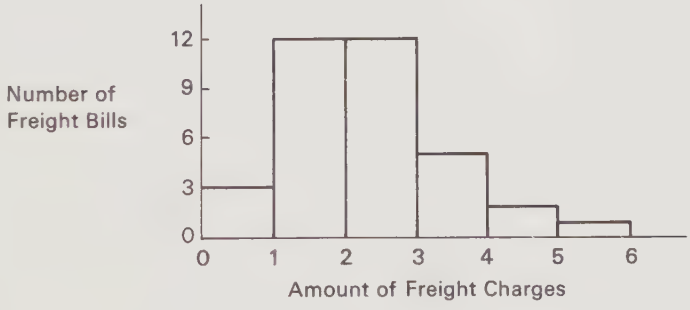



**Figure 2.2**

Answers

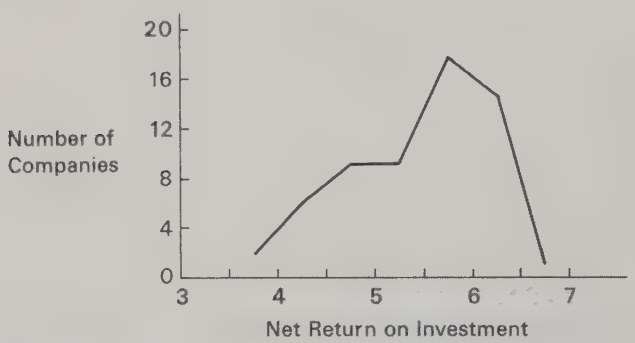
1.

| Net Return on Investment | Number of Companies | Per Cent of Companies |
|---|---|---|
| 3.50–3.99 | 2 | 3 |
| 4.00–4.49 | 6 | 10 |
| 4.50–4.99 | 9 | 15 |
| 5.00–5.49 | 9 | 15 |
| 5.50–5.99 | 18 | 30 |
| 6.00–6.49 | 15 | 25 |
| 6.50–6.99 | 1 | 2 |
| Total | 60 | 100 |

2.

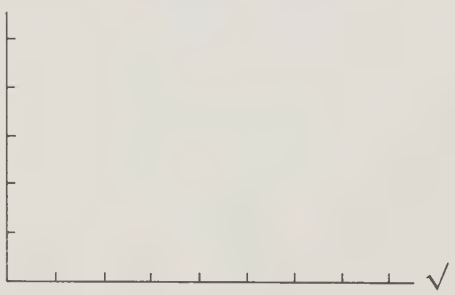
| Amount of Freight Charges | Cumulative Number of Freight Bills | Cumulative Per Cent of Freight Bills |
|---|---|---|
| 0.00 or more | 35 | 100 |
| 1.00 or more | 32 | 91 |
| 2.00 or more | 20 | 57 |
| 3.00 or more | 8 | 23 |
| 4.00 or more | 3 | 9 |
| 5.00 or more | 1 | 3 |
| 6.00 or more | 0 | 0 |

3.




**Problem 4**

Construct the histogram for the frequency distribution in Problem 1.



Example 5 To construct an *ogive*, each cumulative frequency is plotted corresponding to its "more than" or "less than" value. The plotted points are then connected by a series of straight lines. Ogives may be plotted for both absolute cumulative frequencies and per cent cumulative frequencies. Shown in Figure 2.3 is the ogive of the per cent "less than" cumulative frequency distribution developed in Example 2.

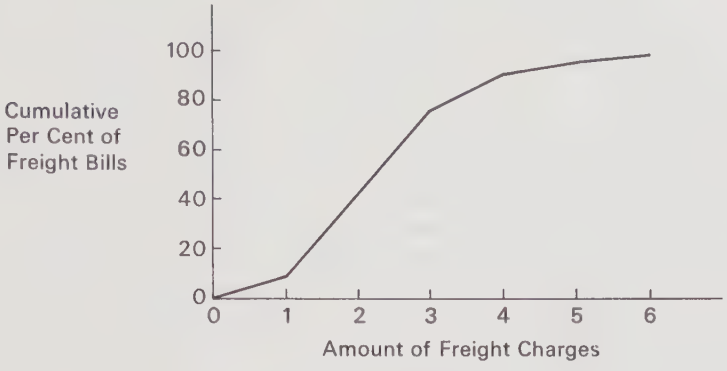
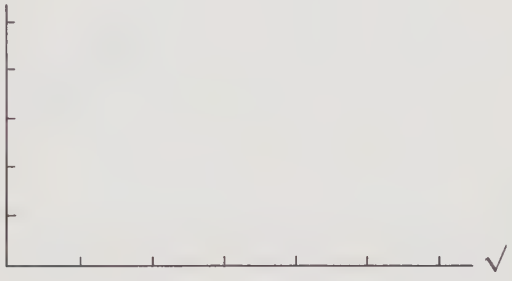



**Figure 2.3**

**Problem 5**

(a) Construct the ogive corresponding to the per cent "more than" cumulative frequency distribution developed in Problem 2.



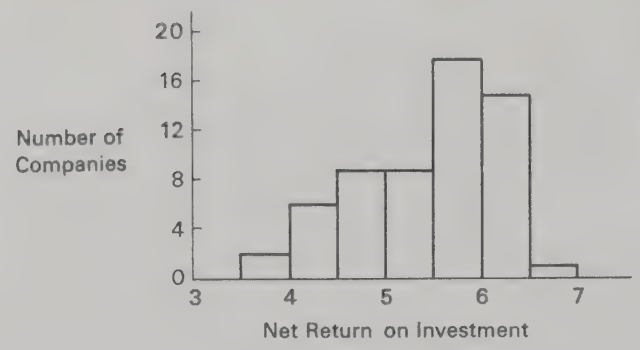
(b) (i) From your ogive, approximately what per cent of the freight bills have charges of \$3.50 or more? _____

(ii) From the original data in Example 1, what is the exact per cent of freight bills having charges of \$3.50 or more? _____

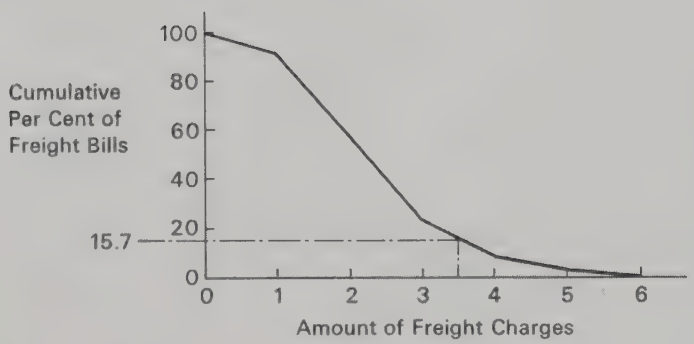
(c) Why is there a discrepancy between the two percentages in (b)? _____

**Answers**

**4.**




**5. (a)**




**(b) (i) Reading from the ogive, 15.7% of the freight bills have charges of $3.50 or more.**

**(ii) From the original data, we see that exactly (4/35) × 100 = 11.4% of the freight bills have charges of $3.50 or more.**

**(c) Connecting the cumulative frequencies by straight lines on the ogive is equivalent to interpolating for values in between. Hence, the ogive provides only an approximation to the actual per cent.**

## REVIEW PROBLEMS

If you were a bit shaky in doing the problems of this section, work out the following review problems. Otherwise, go on to Section 3.

1. Below are given weekly earnings in each of 56 cities across the country.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 78.70 | 86.96 | 103.32 | 94.86 | 105.94 | 98.12 | 106.26 | 86.19 |
| 103.00 | 102.84 | 75.02 | 105.22 | 109.32 | 92.11 | 99.36 | 91.82 |
| 91.14 | 84.62 | 84.43 | 98.07 | 103.87 | 85.48 | 131.79 | 91.15 |
| 83.78 | 92.80 | 87.24 | 105.68 | 117.45 | 128.10 | 122.86 | 94.47 |
| 79.66 | 86.32 | 101.92 | 110.45 | 104.42 | 96.04 | 105.64 | 97.93 |
| 86.00 | 81.38 | 101.77 | 87.91 | 114.35 | 96.19 | 106.23 | 109.08 |
| 119.53 | 113.74 | 105.06 | 121.81 | 93.49 | 96.76 | 100.13 | 96.90 |

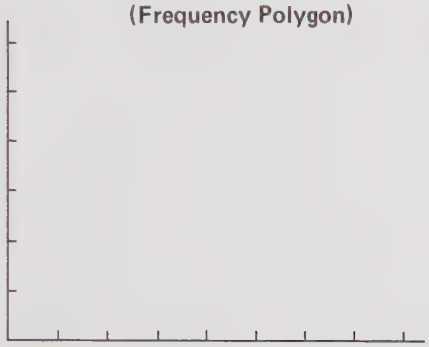
(a) Construct a frequency distribution for these amounts, using the classes shown in Table 2.5.

**Table 2.5**

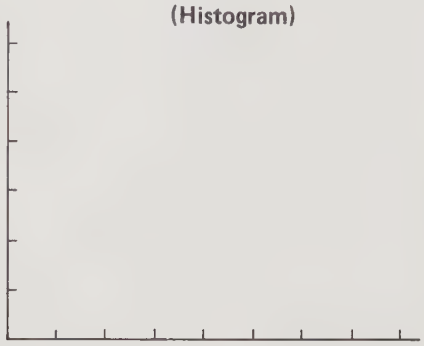
| Weekly Earnings | Number of Cities |
|---|---|
| 70.00– 79.99 | _____ |
| 80.00– 89.99 | _____ |
| 90.00– 99.99 | _____ |
| 100.00–109.99 | _____ |
| 110.00–119.99 | _____ |
| 120.00–129.99 | _____ |
| 130.00–139.99 | _____ |
| Total | _____ √ |

(b) Construct the frequency polygon and histogram of this frequency distribution.

**(Frequency Polygon)**



**(Histogram)**

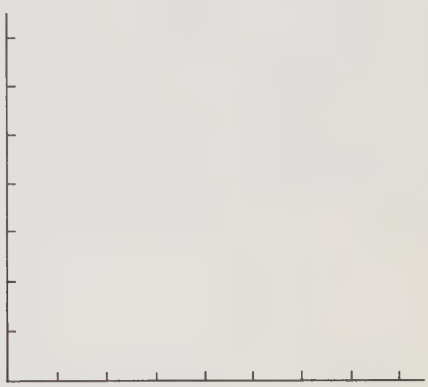


√

2. (a) Construct the absolute "less than" cumulative frequency distribution from the frequency distribution obtained in Review Problem 1.

(b) Construct the ogive of this cumulative frequency distribution.

**(Cumulative Frequency Distribution)** **(Ogive)**



√

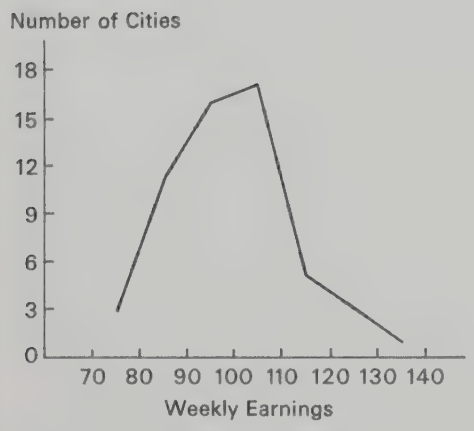
(c) Reading from the ogive, weekly earnings in one half of the cities were below what amount? _____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a)

| Weekly Earnings | Number of Cities |
|---|---|
| 70.00– 79.99 | 3 |
| 80.00– 89.99 | 11 |
| 90.00– 99.99 | 16 |
| 100.00–109.99 | 17 |
| 110.00–119.99 | 5 |
| 120.00–129.99 | 3 |
| 130.00–139.99 | 1 |
| Total | 56 |

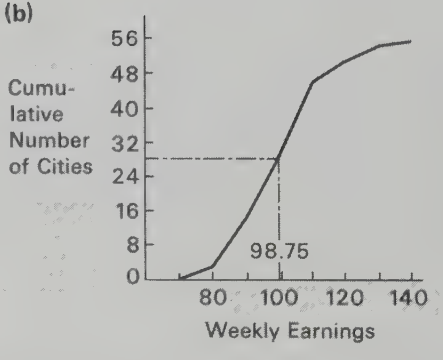
(b)



2. (a)

| Weekly Earnings | Number of Cities |
|---|---|
| Less than 70.00 | 0 |
| Less than 80.00 | 3 |
| Less than 90.00 | 14 |
| Less than 100.00 | 30 |
| Less than 110.00 | 47 |
| Less than 120.00 | 52 |
| Less than 130.00 | 55 |
| Less than 140.00 | 56 |

(b)




(c) From the ogive, you should obtain an answer of $98.75. From the original data, it is seen that one half of the cities had weekly earnings less than or equal to $98.07.

# 3

# Characteristics of Frequency Distributions: Averages

In this section, we consider different types of averages used to summarize the location of a frequency distribution.

**A. Summation Notation.** Several formulas of this section will use the symbol $\Sigma$ (Greek letter, capital sigma) to designate summation. The three expressions in which $\Sigma$ will be used are:

$$\text{(i)} \quad \Sigma X = X_1 + X_2 + \cdots + X_n,$$
$$\text{(ii)} \quad (\Sigma X)^2 = (X_1 + X_2 + \cdots + X_n)^2,$$
$$\text{and (iii)} \quad \Sigma X^2 = X_1^2 + X_2^2 + \cdots + X_n^2.$$

Example 1 If $X_1 = 3$, $X_2 = 1$, and $X_3 = 7$, then

$$\Sigma X = X_1 + X_2 + X_3 = 3 + 1 + 7 = 11,$$
$$(\Sigma X)^2 = (X_1 + X_2 + X_3)^2 = (3 + 1 + 7)^2 = 121,$$
$$\text{and } \Sigma X^2 = X_1^2 + X_2^2 + X_3^2 = 3^2 + 1^2 + 7^2 = 59.$$

**Problem 1**

Let $X_1 = 4, X_2 = 5, X_3 = 10$, and $X_4 = 1$. Calculate $\Sigma X =$ _____, $(\Sigma X)^2 =$ _____, and $\Sigma X^2 =$ _____, for these four $X$ values. √

**B. Population and Sample.** The aggregate or totality of items under consideration in a specific problem is called the population under study. A sample is a part of the population under study selected so that inferences can be made from the sample about the entire population. For instance, the 1350 purchase orders of a manufacturing firm issued in June may represent the population under study. Fifty of these purchase orders selected at random would represent a sample from this population.

A given set of items may be considered a population for one study and a sample for another study. Thus, a shipment of 100 units from a manufacturer would be considered a population in making a decision to accept or reject that shipment. But if we are interested in information about the quality of the manufacturer's production process, the same shipment would be regarded as a sample from the process.

In this and the next section, we assume that we are dealing with sample data.

**C. Arithmetic Mean, Ungrouped Sample Data.** The arithmetic mean is defined as the sum of the values of the items divided by the number of items. The arithmetic mean of a sample, denoted by $\bar{X}$ (read: $X$ bar), may be written symbolically as:

$$\bar{X} = \frac{\Sigma X}{n}, \quad \text{where } n \text{ is the number of items.}$$

**Answers**

1. $\Sigma X = 20$, $(\Sigma X)^2 = 400$, $\Sigma X^2 = 142$.

Example 2 Consider the $X$ values: 15, 7, 11, 4, and 8. The arithmetic mean of these five values is:

$$\overline{X} = \frac{15 + 7 + 11 + 4 + 8}{5} = \frac{45}{5} = 9$$

**Problem 2**

(a) Find the arithmetic mean of the following values: 4, 8, 3, 0, 11, 12, and 4. _____
(b) If the observations in (a) are in dollars, in what units is the mean expressed? _____
(c) What is the sum of the seven values? _____ How can this sum be obtained from the arithmetic mean? _____ √

**D. Median, Ungrouped Sample Data.** The median of a group of sample items is the value of the middle item when the items are arranged in ascending or descending order of magnitude. The middle item is the $(n + 1)/2$ item.

Example 3 Consider the five $X$ values in Example 2. Arranging these values in ascending order gives: 4, 7, 8, 11, 15. The middle item is the $(5 + 1)/2$ = third item. The value of the third item is 8. Hence, the median equals 8.

**Problem 3**

Consider the seven $X$ values in Problem 2.
(a) Determine the median value. _____
(b) Is the median equal to the arithmetic mean of these same seven values? _____
(c) Are the mean and median the same when the $X$ values are 1, 3, 7, 11, 13? _____ What does this suggest about the conditions when the mean and median are the same? _____ _____ √

**Problem 4**

Again, consider the seven $X$ values in Problem 2. For these seven values, $\overline{X} = 6$ and median = 4.
(a) Fill in the blanks in Table 3.1 and sum the two sets of deviations.

**Table 3.1**

| $X$ | $X - \overline{X}$ | $X$ − median |
|---|---|---|
| 4 | 4 − 6 = −2 | 4 − 4 = 0 |
| 8 | 8 − 6 = +2 | 8 − 4 = +4 |
| 3 | 3 − 6 = −3 | 3 − 4 = −1 |
| 0 | _____ | _____ |
| 11 | _____ | _____ |
| 12 | _____ | _____ |
| 4 | _____ | _____ |
| | $\Sigma (X - \overline{X})$ = _____ | $\Sigma$ ($X$ − median) = _____ √ |

(b) From the results in (a), what general statement can be made about $\Sigma(X - \overline{X})$? _____ _____

Does this generalization also apply to $\Sigma(X - \text{median})$?_____

_____ √

Example 4 When there are an even number of items, the expression "middle item" in the definition of the median is ambiguous. In this case, the median is set equal to the average value of the middle two items when all the values are arranged according to magnitude.

Consider the following six $X$ values: 3, 8, 3, 4, 6, and 12. Arranging these values in descending order gives: 12, 8, 6, 4, 3, 3. The middle item is the $(6 + 1)/2 = 3.5$ item, and the middle two items are the third and fourth items. The values of the middle two items are 6 and 4. The average of 6 and 4 is: $(6 + 4)/2 = 5$. Hence, the median of the six values is equal to 5.

**Problem 5**

Consider the following eight $X$ values: 113, 43, 7, 18, 83, 21, 16, and 76. What is the median of these values? _____ √

**E. Geometric Mean, Ungrouped Sample Data.** Given $n$ sample items, the geometric mean of these items is the $n$th root of the product of their values. Symbolically, the geometric mean is:

$$\text{geometric mean} = \sqrt[n]{X_1 X_2 \cdots X_n} .$$

Because the arithmetic involved in finding the $n$th root is complex, the following logarithmic form of the geometric mean usually is used:

$$\log_{10}(\text{geometric mean}) = \frac{\log_{10} X_1 + \log_{10} X_2 + \cdots + \log_{10} X_n}{n}.$$

Example 5 Consider the five $X$ values in Example 2. To obtain the geometric mean of these values, we first obtain their logarithms using Table A-10. Next, we sum the logarithms and divide by the number of $X$ values, five in this case. The resulting value is the logarithm of the geometric mean. The geometric mean is the antilogarithm of this value.

| $X$ | $\log_{10} X$ |
|---|---|
| 15 | 1.17609 |
| 7 | .84510 |
| 11 | 1.04139 |
| 4 | .60206 |
| 8 | .90309 |
| Total | 4.56773 |

$$\log_{10}(\text{geometric mean}) = \frac{4.56773}{5} = .91355;$$

$$\text{geometric mean} = \text{antilog}_{10}(.91355) = 8.195.$$

**Problem 6**

Obtain the geometric mean of the following three $X$ values: 103, 10, and 25._____

_____ √

**F. Weighed Arithmetic Mean, Ungrouped Sample Data.** At times, we need an arithmetic mean where the items to be averaged require different weights.

Example 6 The per hundred weight prices that a rancher received for his cattle, in the years 1966, 1967, and 1968, were \$25.00, \$23.00, and \$29.00 respectively. To obtain the mean price per hundred weight that he received over the three years, it is necessary to weight the prices by the number of hundred weight sold in each year. The appropriate calculation is:

Answers

2. (a) $\bar{X} = 6$.

   (b) Dollars.

   (c) Sum = 42. The sum is obtained by multiplying $\bar{X}$ by $n$.

3. (a) Median = 4.

   (b) No, $\bar{X} = 6$ and median = 4.

   (c) Yes, $\bar{X} = 7$ and median = 7. The equality of the mean and median suggests that when the $X$ values are placed in balanced positions around the median, the mean and median are the same.

4. (a)

| $X$ | $X - \bar{X}$ | $X$ − median |
|---|---|---|
| 4 | −2 | 0 |
| 8 | +2 | +4 |
| 3 | −3 | −1 |
| 0 | −6 | −4 |
| 11 | +5 | +7 |
| 12 | +6 | +8 |
| 4 | −2 | 0 |
| | $\Sigma(X - \bar{X}) = 0$ | $\Sigma(X - \text{median}) = +14$ |

   (b) $\Sigma(X - \bar{X}) = 0$. The sum of the deviations about the mean is always zero. $\Sigma(X - \text{median}) \neq 0$. In general, the sum of the deviations about the median is not zero.

5. 7, 16, 18, 21, 43, 76, 83, 113: median = (21 + 43)/2 = 32.

6.

| $X$ | $\log_{10} X$ |
|---|---|
| 103 | 2.01284 |
| 10 | 1.00000 |
| 25 | 1.39794 |
| Total | 4.41078 |

$\log_{10}$ (geometric mean) = 4.41078/3 = 1.47026;
geometric mean = 29.5.

Table 3.2

| Year | Price (per hundred weight) | Quantity (in hundred weight) | Price x Quantity |
|---|---|---|---|
| 1966 | 25.00 | 8,000 | 200,000 |
| 1967 | 23.00 | 10,000 | 230,000 |
| 1968 | 29.00 | 4,000 | 116,000 |
| Total | | 22,000 | 546,000 |

Weighted mean price per hundred weight = 546,000/22,000 = $24.82.

Problem 7

Ten per cent of the 1000 items in lot A are defective. Five per cent of the 10,000 items in lot B are defective. Five per cent of the 5000 items in lot C are defective. Find the per cent of defective items in all three lots combined. _____ √

**G. Arithmetic Mean, Grouped Sample Data.** The calculation of the mean of a frequency distribution may be represented symbolically as follows:

$$\overline{X} = \frac{\Sigma fX}{n},$$

where $X$ represents a class midpoint, $f$ is the frequency of that class, and $n = \Sigma f$ is the number of items in the distribution.

Example 7 Shown in Table 3.3 is the frequency distribution of the net return on investment in 1968 for 60 life insurance companies. The midpoint of the first class is (3.50 + 4.00)/2 = 3.75, and similarly for the other classes. If the first class had been stated "3.50–under 4.00," the midpoint would still be 3.75. In either case, the class width is considered to be 4.00–3.50 = .50. The mean net return on investment is calculated as follows:

**Table 3.3**

| Net Return on Investment | | Number of Companies | |
|---|---|---|---|
| Class | Midpoint $X$ | $f$ | $fX$ |
| 3.50–3.99 | 3.75 | 2 | 7.50 |
| 4.00–4.49 | 4.25 | 6 | 25.50 |
| 4.50–4.99 | 4.75 | 9 | 42.75 |
| 5.00–5.49 | 5.25 | 9 | 47.25 |
| 5.50–5.99 | 5.75 | 18 | 103.50 |
| 6.00–6.49 | 6.25 | 15 | 93.75 |
| 6.50–6.99 | 6.75 | 1 | 6.75 |
| | | $n = \Sigma f = 60$ | $\Sigma fX = 327.00$ |

$$\overline{X} = 327.00/60 = 5.45.$$

Hence the mean net return on investment for the 60 companies is 5.45%.

**Problem 8**

Calculate the mean of the following frequency distribution:

**Table 3.4**

| Class | $X$ | $f$ | $fX$ |
|---|---|---|---|
| 0.00–0.99 | | 15 | |
| 1.00–1.99 | | 12 | |
| 2.00–2.99 | | 3 | |
| 3.00–3.99 | | 2 | |
| 4.00–4.99 | | 2 | |
| 5.00–5.99 | | 1 | |
| Total | | 35 | |

$\overline{X} =$ _____ √

**H. Median, Grouped Sample Data.** The location of the middle item in grouped data is given by $n/2$. The value of the middle item of a frequency distribution, the median, is obtained from:

$$\text{median} = L + \frac{n_1}{n_2}(i),$$

where $L$ is the lower limit of the median class, $n_1$ is the number of frequencies that must be covered in the median class in order to reach the middle item, $n_2$ is the frequency of the median class, and $i$ is the width of this class.

Answers

7.

| Lot | Per Cent Defective | Lot Size | Per Cent Defective X Lot Size |
|---|---|---|---|
| A | 10 | 1,000 | 10,000 |
| B | 5 | 10,000 | 50,000 |
| C | 5 | 5,000 | 25,000 |
| Total | | 16,000 | 85,000 |

Weighted mean per cent defective = 85,000/16,000 = 5.31%.

8.

| $X$ | $f$ | $fX$ |
|---|---|---|
| 0.50 | 15 | 7.50 |
| 1.50 | 12 | 18.00 |
| 2.50 | 3 | 7.50 |
| 3.50 | 2 | 7.00 |
| 4.50 | 2 | 9.00 |
| 5.50 | 1 | 5.50 |
| Total | 35 | 54.50 |

$$\overline{X} = \frac{54.50}{35} = 1.56.$$

Example 8 Consider the frequency distribution in Example 7. For that frequency distribution, the location of the middle item is 60/2 = 30. Cumulating the frequencies, we find the median class:

Table 3.5

| Class | Frequency | Cumulative frequency |
|---|---|---|
| 3.50–3.99 | 2 | 2 |
| 4.00–4.49 | 6 | 8 |
| 4.50–4.99 | 9 | 17 |
| 5.00–5.49 | 9 | 26 |
| 5.50–5.99 | 18 | 44 ← thirtieth item is in this class |

We thus have:

middle item: $60 \times 1/2 = 30$,
median class: $5.50 - 5.99$,
$L = 5.50$,
$n_1 = 30-2-6-9-9 = 4$,
$n_2 = 18$,
$i = .50$.

Thus the median net return on investment equals:

$$\text{median} = 5.50 + \frac{4}{18}(.50) = 5.61\%.$$

**Problem 9**

(a) Determine the median of the frequency distribution in Problem 8. _____

(b) How does the median compare with the mean? (See Problem 8.) _____
What does this imply about the symmetry of the frequency distribution? _____
_____ √

**I. Percentiles, Grouped Sample Data.** The $p$th percentile of a frequency distribution is that value such that $p$ per cent of the items are less than it. The location of the $p$th percentile is given by $n(p/100)$. To obtain the value of the $p$th percentile, we again use the formula:

$$p\text{th percentile} = L + \frac{n_1}{n_2}(i),$$

where $L$ is the lower limit of the $p$th percentile class, $n_1$ is the number of frequencies that must be covered in this class in order to reach the $p$th percentile item, $n_2$ is the frequency of the $p$th percentile class, and $i$ is the width of this class.

Example 9 To find the 20th percentile of the frequency distribution in Example 7, we observe that:

20th percentile item: $60 \times .20 = 12$,

20th percentile class: $4.50 - 4.99$,

$L = 4.50$,

$n_1 = 12 - 2 - 6 = 4$,

$n_2 = 9$,

$i = .50$.

Hence: 20th percentile $= 4.50 + \frac{4}{9}(.50) = 4.72\%$.

**Problem 10**

Find the 80th percentile of the frequency distribution in Problem 8. _____ √

Example 10 Several common percentiles are known by other names. For example:

| | |
|---|---|
| 25th percentile | first quartile |
| 50th percentile | median or second quartile |
| 75th percentile | third quartile |

Answers

**9.** (a) Middle item: $35 \times 1/2 = 17.5$,
median class: $1.00 - 1.99$,
$L = 1.00$,
$n_1 = 17.5 - 15 = 2.5$,
$n_2 = 12$,
$i = 1.00$.

$$\text{median} = 1.00 + \frac{2.5}{12}(1.00) = 1.21.$$

(b) **The mean and the median are not the same. This indicates that the frequency distribution is not symmetrical, in other words, the observations are not placed in balanced positions around the median.**

**10.** 80th percentile item: $35 \times .80 = 28$,
80th percentile class: $2.00 - 2.99$,
$L = 2.00$,
$n_1 = 28 - 15 - 12 = 1$,
$n_2 = 3$,
$i = 1.00$,

$$\text{80th percentile} = 2.00 + \frac{1}{3}(1.00) = 2.33.$$

## REVIEW PROBLEMS

1. Consider the following $X$ values: 100, 10, 24, 6. For these values, find:
   (a) arithmetic mean _____ ;
   (b) median _____ ;
   (c) geometric mean _____ .√

2. The ages, to the nearest birthday, of 50 boys recruited into a door-to-door magazine selling campaign are distributed as follows:

| Age, to Nearest Birthday | Number of Boys |
|---|---|
| 15 | 4 |
| 16 | 10 |
| 17 | 10 |
| 18 | 20 |
| 19 | 6 |
| Total | 50 |

   (a) What are the upper and lower class limits for the class designated by 15? _____ √
   (b) Calculate:
      (i) mean age _____ ;
      (ii) median age _____ ;
      (iii) first quartile age _____ .√

## ANSWERS TO REVIEW PROBLEMS

1. (a) 35.
   (b) 17.
   (c)

| $X$ | $\log_{10} X$ |
|---|---|
| 100 | 2.00000 |
| 10 | 1.00000 |
| 24 | 1.38021 |
| 6 | .77815 |
| Total | 5.15836 |

$\log_{10}$(geometric mean) = 5.15836/4 = 1.28959;
geometric mean = 19.5.

2. (a) 14.5—under 15.5.
   (b) (i)

| $X$ | $f$ | $fX$ |
|---|---|---|
| 15 | 4 | 60 |
| 16 | 10 | 160 |
| 17 | 10 | 170 |
| 18 | 20 | 360 |
| 19 | 6 | 114 |
| Total | 50 | 864 |

$$\overline{X} = \frac{864}{50} = 17.28.$$

(ii) Middle item: 50 X 1/2 = 25,
median class: 17.5—under 18.5 (18 is the midpoint),
$L$ = 17.5,
$n_1 = 25 - 4 - 10 - 10 = 1$,
$n_2$ = 20,
$i$ = 1.

$$\text{median} = 17.5 + \frac{1}{20}(1) = 17.55.$$

(iii) First quartile item: 50 X 1/4 = 12.5,
first quartile class: 15.5—under 16.5 (16 is the midpoint),
$L$ = 15.5,
$n_1 = 12.5 - 4 = 8.5$,
$n_2$ = 10,
$i$ = 1.

$$\text{first quartile} = 15.5 + \frac{8.5}{10}(1) = 16.35.$$

# 4

# Characteristics of Frequency Distributions: Measures of Dispersion and Skewness

In this section, we consider various measures of dispersion and skewness. We assume, as in the previous section, that the data are from a sample.

**A. Range, Ungrouped Sample Data.** The range is the difference between the largest and the smallest values in the group of items.

Example 1 In a weaving mill, seven observations were taken at random times on a section of 400 looms to determine the number of looms working at a given time. The results were: 368, 353, 376, 361, 366, 351, 359. The range of the seven observations is $376 - 351 = 25$ looms.

**Problem 1**

An insurance company recorded the times required to process six fire insurance claims. The times (in days) are: 1, 6, 4, 4, 12, 3.

(a) Calculate the range of these six values. _____

(b) In what units is the range expressed? _____ √ (For answers, see page 30.)

**B. Mean Deviation, Ungrouped Sample Data.** The mean deviation, also called the average deviation, is the sum of the absolute deviations of the items from their arithmetic mean, divided by the number of items. In notation:

$$\text{mean deviation} = \frac{\Sigma |X - \bar{X}|}{n}.$$

The two vertical lines enclosing $X - \bar{X}$ signify that minus signs are ignored, that is, we take the absolute value of each deviation.

Example 2 Consider the seven values in Example 1. The mean deviation of these values may be calculated as follows:

Table 4.1

| $X$ | $X - \bar{X}$ | $\lvert X - \bar{X} \rvert$ |
|---|---|---|
| 368 | +6 | 6 |
| 353 | −9 | 9 |
| 376 | +14 | 14 |
| 361 | −1 | 1 |
| 366 | +4 | 4 |
| 351 | −11 | 11 |
| 359 | −3 | 3 |
| $\Sigma X = 2534$ | $\Sigma(X - \bar{X}) = 0$ | $\Sigma\lvert X - \bar{X} \rvert = 48$ |
| $\bar{X} = 2534/7 = 362.$ | | mean deviation $= 48/7 = 6.86.$ |

Note that as a check on your work, the sum of the deviations about the mean should be zero.

**Problem 2**

Find the mean deviation of the six values in Problem 1. _____√

**C. Standard Deviation, Ungrouped Sample Data.** The standard deviation is obtained by summing the squared deviations about the arithmetic mean, dividing by $n - 1$, and taking the positive square root of the resulting value. In notation:

$$s = \sqrt{\frac{\Sigma(X - \bar{X})^2}{n - 1}},$$

where $s$ is the sample standard deviation. The square of the standard deviation is called the *variance*, and is denoted $s^2$. (It should be noted that in some books the formula for $s$ is written with a denominator $n$, rather than $n - 1$.)

Example 3 Consider the seven values in Example 1. The standard deviation of these seven values may be calculated as follows:

Table 4.2

| $X$ | $X - \bar{X}$ | $(X - \bar{X})^2$ |
|---|---|---|
| 368 | +6 | 36 |
| 353 | −9 | 81 |
| 376 | +14 | 196 |
| 361 | −1 | 1 |
| 366 | +4 | 16 |
| 351 | −11 | 121 |
| 359 | −3 | 9 |
| | 0 | $\Sigma(X - \bar{X})^2 = 460$ |
| $\bar{X} = 362$ | | |

$$s = \sqrt{\frac{460}{7-1}} = \sqrt{76.66} = 8.76.$$

The variance is 76.66.

**Problem 3**

(a) Find the standard deviation and variance of the six values in Problem 1.

$s =$ _____ ,

$s^2 =$ _____ .

(b) In what units is $s$ expressed? _____ √

**D. Standard Deviation, Grouped Sample Data.** The calculation of the standard deviation of a frequency distribution may be represented symbolically as follows:

$$s = \sqrt{\frac{\Sigma f(X-\overline{X})^2}{n-1}} .$$

Here $X$ is a class midpoint, $f$ is the frequency of that class, $\overline{X}$ is the arithmetic mean of the distribution, and $n = \Sigma f$ is the number of items in the distribution.

Example 4 Shown in Table 4.3 is the net return on investment in 1968 for 60 life insurance companies. The standard deviation of the net return on investment is calculated as follows:

**Table 4.3**

| Net Return on Investment: Class | Midpoint $X$ | Number of Companies $f$ | Deviation $X-\overline{X}$ | Squared Deviation $(X-\overline{X})^2$ | Class Total of Squared Deviations $f(X-\overline{X})^2$ |
|---|---|---|---|---|---|
| 3.50–3.99 | 3.75 | 2 | −1.70 | 2.89 | 5.78 |
| 4.00–4.49 | 4.25 | 6 | −1.20 | 1.44 | 8.64 |
| 4.50–4.99 | 4.75 | 9 | −0.70 | 0.49 | 4.41 |
| 5.00–5.49 | 5.25 | 9 | −0.20 | 0.04 | 0.36 |
| 5.50–5.99 | 5.75 | 18 | 0.30 | 0.09 | 1.62 |
| 6.00–6.49 | 6.25 | 15 | 0.80 | 0.64 | 9.60 |
| 6.50–6.99 | 6.75 | 1 | 1.30 | 1.69 | 1.69 |
| Total | | 60 | | | 32.10 |

$\overline{X} = 5.45$ (from Section 3, Example 7);

$$s = \sqrt{\frac{32.10}{59}} = \sqrt{.544} = .74.$$

**Problem 4**

Calculate the standard deviation of the following frequency distribution. For this frequency distribution, $\overline{X} = 1$.

**Table 4.4**

| Midpoint $X$ | Frequency $f$ |
|---|---|
| 0 | 13 |
| 1 | 0 |
| 2 | 1 |
| 3 | 6 |
| Total | 20 |

$s =$ _____ √

(For answer, see page 32.)

Answers

1. (a) Range = 12 − 1 = 11 days.
   (b) Days.

2.

| $X$ | $X - \bar{X}$ | $\lvert X - \bar{X} \rvert$ |
|---|---|---|
| 1 | −4 | 4 |
| 6 | +1 | 1 |
| 4 | −1 | 1 |
| 4 | −1 | 1 |
| 12 | +7 | 7 |
| 3 | −2 | 2 |
| Total 30 | 0 | 16 |

$\bar{X} = 5$. mean deviation = 16/6 = 2.67.

3. (a)

| $X$ | $X - \bar{X}$ | $(X - \bar{X})^2$ |
|---|---|---|
| 1 | −4 | 16 |
| 6 | +1 | 1 |
| 4 | −1 | 1 |
| 4 | −1 | 1 |
| 12 | +7 | 49 |
| 3 | −2 | 4 |
| Total 30 | 0 | 72 |

$\bar{X} = 5$.

$$s = \sqrt{\frac{72}{6-1}} = \sqrt{14.4} = 3.79; \quad s^2 = 14.4.$$

(b) Days.

**E. Interquartile Range, Grouped Sample Data.** The range between the first and third quartiles of a frequency distribution is called the interquartile range.

**Problem 5**

For the frequency distribution in Example 4, the first and third quartiles are 4.89 and 6.03 respectively.

(a) Calculate the interquartile range. _____
(b) Approximately what per cent of the firms have rates of return in the interval 4.89 to 6.03? _____
Why is this percentage an approximation? _____
_____ √

**F. Coefficient of Variation.** The coefficient of variation is obtained by expressing the standard deviation as a per cent of the mean, as follows:

$$\text{coefficient of variation} = \frac{s}{\bar{X}} \cdot 100.$$

While the standard deviation, $s$, measures *absolute* variability in the data, the coefficient of variation measures *relative* variability (variability relative to the mean).

Example 5 A filling machine fills two sizes of cans, "large" and "small." Samples of 30 filled cans of each size were taken and the weight of fill in each can was determined. The means and standard deviations of the two samples are shown in Table 4.5. The calculation of the coefficient of variation is shown for the sample of large cans.

**Table 4.5**

| Can Size | $\overline{X}$ | $s$ | Coefficient of Variation |
|---|---|---|---|
| Large | 25 ounces | .375 ounces | (.375/25) x 100 =1.5% |
| Small | 16 ounces | .320 ounces | _____ |

**Problem 6**

(a) Calculate the coefficient of variation for the sample of small cans in Example 5.
(b) What are the units of the coefficient of variation? _____
(c) For which size of can is the absolute variability in fill greater? _____
Relative variability? _____ √

**G. Pearson's Coefficient of Skewness.** Pearson's coefficient of skewness is defined symbolically as:

$$\text{skewness} = \frac{3(\overline{X} - \text{median})}{s}.$$

When the frequency distribution under consideration is skewed to the left, as in the Figure 4.1a, Pearson's coefficient of skewness is negative. When the frequency distribution is skewed to the right, as in the Figure 4.1b, the coefficient is positive. When the frequency distribution is symmetrical, the coefficient is zero.

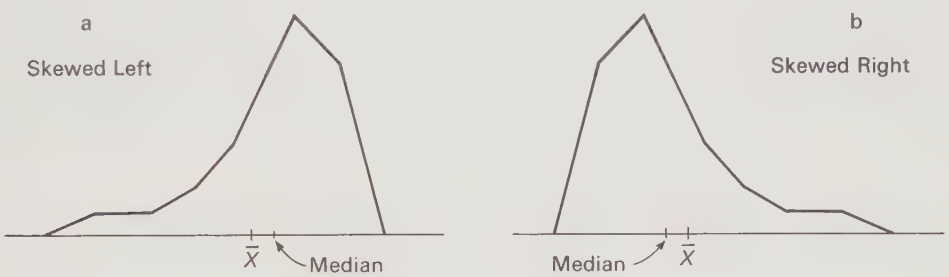



**Figure 4.1**

Example 6 Consider the frequency distribution in Example 4. The mean and median of the distribution are 5.45 and 5.61 respectively. (See Section 3, Examples 7 and 8 respectively.) In Example 4, it was seen that $s = .74$. Thus, for this frequency distribution, Pearson's coefficient of skewness equals:

$$\text{skewness} = \frac{3(5.45 - 5.61)}{.74} = -.65.$$

The frequency distribution is therefore skewed to the left. The frequency polygon of this frequency distribution is shown in Section 2, Problem 3.

**Problem 7**

A state promotion brochure states that the mean income of families in the state is $5400. It is known that the frequency distribution of family incomes is skewed to the right, similar to the Figure 4.1b. On the basis of this information, is it valid to say that less than half the families in the state have incomes in excess of $5400? Why?

_____ √

**Answers**

4.

| $X$ | $f$ | $X-\bar{X}$ | $(X-\bar{X})^2$ | $f(X-\bar{X})^2$ |
|---|---|---|---|---|
| 0 | 13 | −1 | 1 | 13 |
| 1 | 0 | 0 | 0 | 0 |
| 2 | 1 | +1 | 1 | 1 |
| 3 | 6 | +2 | 4 | 24 |
| Total | 20 | | | 38 |

$$s = \sqrt{\frac{38}{20-1}} = \sqrt{2} = 1.414.$$

5. (a) 6.03 − 4.89 = 1.14.

   (b) 75 − 25 = 50%. The reason why 50% is approximate is that we interpolated to find the first and third quartiles.

6. (a) (.320/16)100 = 2%.

   (b) Since $s$ and $\bar{X}$ are expressed in the same units, the ratio $s/\bar{X}$ is unitless. Multiplying this ratio by 100 gives the coefficient of variation, expressed in per cent.

   (c) The large cans exhibit greater absolute variability because the standard deviation for large cans is greater than for small cans. The small cans exhibit greater relative variability because the coefficient of variation for small cans is greater than for large cans.

7. Statement is valid. Since the skewness is positive, or to the right, the formula for Pearson's coefficient of skewness implies that the mean is to the right of the median on the income scale. Since the median corresponds to the 50th percentile, and the mean is higher than the median, it follows that less than half the families have incomes greater than the mean income.

**Problem 8**

Refer to Problem 7. Assume the median family income in the state is $4800, and the standard deviation of family incomes in the state is $3000. Calculate Pearson's coefficient of skewness for the frequency distribution of family incomes. _____ √

**H. Population Mean and Population Standard Deviation.** Thus far, the discussion of characteristics of frequency distributions has related to sample data only. If one is dealing with the totality of items under study, that is, the entire population, the mean of the items is the population mean, denoted by $\mu$ (Greek mu), and the standard deviation of the items is the population standard deviation, denoted by $\sigma$ (Greek sigma).

For a population with a finite number of items, the population mean and the population standard deviation are defined as follows:

$$\mu = \frac{\Sigma X}{N},$$

$$\sigma = \sqrt{\frac{\Sigma(X-\mu)^2}{N}},$$

where $N$ is the finite population size. The computation of $\mu$ and $\sigma$ for a finite population is analogous to the computation of the sample characteristics $\bar{X}$ and $s$ respectively (except that the denominator of $\sigma$ is $N$).

For populations which contain an infinite number of items, one can also define a population mean $\mu$ and a population standard deviation $\sigma$, but the calculations require mathematical methods beyond the scope of this workbook.

## REVIEW PROBLEMS

1. Five bids were made for a highway construction contract. The amounts of the bids (in $ thousands) were 100, 112, 108, 98, and 102. For the five bid amounts, determine their:
   (a) range _____
   (b) mean deviation _____
   (c) standard deviation _____
   (d) coefficient of variation _____ √

2. The EZ Vacuum Cleaner Company recorded the number of warranty service calls required on 26 new cleaners. The following frequency distribution was obtained:

**Table 4.6**

| Number of Calls | Number of Cleaners |
|---|---|
| 0 | 7 |
| 1 | 0 |
| 2 | 9 |
| 3 | 6 |
| 4 | 4 |
| 5 | 0 |
| Total | 26 |

Determine the standard deviation of this distribution. (*Hint:* Since each class contains only one value, this is automatically the midpoint of the class.) $s$ = _____ √

3. The frequency distribution of the weekly earnings in 56 cities is shown in Table 4.7, together with the values of several measures calculated from this distribution.

**Table 4.7**

| Weekly Earnings | Number of Cities |
|---|---|
| 70.00– 79.99 | 3 |
| 80.00– 89.99 | 11 |
| 90.00– 99.99 | 16 |
| 100.00–109.99 | 17 |
| 110.00–119.99 | 5 |
| 120.00–129.99 | 3 |
| 130.00–139.99 | 1 |
| Total | 56 |

first quartile = 90.00
median = 98.75
third quartile = 107.06
$\overline{X}$ = 99.11
$s$ = 13.04

For this distribution determine:
(a) interquartile range; _____
(b) Pearson's coefficient of skewness. _____ Is this distribution skewed? If so, in what way? _____ √

Answers

8. $\bar{X}$ = \$5400; median = \$4800; $s$ = \$3000.

$$\text{skewness} = \frac{3(5400-4800)}{3000} = +.60.$$

## ANSWERS TO REVIEW PROBLEMS

1. (a) $112 - 98 = 14$.

(b)

| $X$ | $X - \bar{X}$ | $\lvert X - \bar{X}\rvert$ |
|---|---|---|
| 100 | −4 | 4 |
| 112 | +8 | 8 |
| 108 | +4 | 4 |
| 98 | −6 | 6 |
| 102 | −2 | 2 |
| $\Sigma X = 520$ | 0 | $\Sigma\lvert X - \bar{X}\rvert = 24$ |

$\bar{X} = 520/5 = 104.$ mean deviation $= \dfrac{\Sigma\lvert X - \bar{X}\rvert}{n} = 24/5 = 4.8.$

(c) $\bar{X} = 104$. (See the answer to Review Problem 1(b).)

| $X$ | $X - \bar{X}$ | $(X - \bar{X})^2$ |
|---|---|---|
| 100 | −4 | 16 |
| 112 | +8 | 64 |
| 108 | +4 | 16 |
| 98 | −6 | 36 |
| 102 | −2 | 4 |
| Total | 0 | $\Sigma(X - \bar{X})^2 = 136$ |

$$s = \sqrt{\frac{\Sigma(X-\bar{X})^2}{n-1}} = \sqrt{\frac{136}{5-1}} = \sqrt{34} = 5.8.$$

(d) $(5.83/104)100 = 5.6\%$.

2.

| $X$ | $f$ | $fX$ | $X - \bar{X}$ | $(X - \bar{X})^2$ | $f(X - \bar{X})^2$ |
|---|---|---|---|---|---|
| 0 | 7 | 0 | −2 | 4 | 28 |
| 1 | 0 | 0 | −1 | 1 | 0 |
| 2 | 9 | 18 | 0 | 0 | 0 |
| 3 | 6 | 18 | 1 | 1 | 6 |
| 4 | 4 | 16 | 2 | 4 | 16 |
| 5 | 0 | 0 | 3 | 9 | 0 |
| Total | 26 | 52 | | | $\Sigma f(X - \bar{X})^2 = 50$ |

$$\bar{X} = \frac{\Sigma fX}{n} = \frac{52}{26} = 2; \quad s = \sqrt{\frac{\Sigma f(X-\bar{X})^2}{n-1}} = \sqrt{\frac{50}{26-1}} = \sqrt{2} = 1.414.$$

3. (a) $107.06 - 90.00 = 17.06$.

(b) $\dfrac{3(99.11 - 98.75)}{13.04} = +.08$; slightly skewed to the right.

5

# Introduction to Probability

**A. Sample Space.** It is customary to refer to any process for which probability concepts are applicable as a *random experiment.* The different possible results of interest in a random experiment may be classified into *basic outcomes.* The set of all possible basic outcomes constitutes the *sample space* of the random experiment.

Example 1 The performances of three machines during a one-week production period may be considered a random experiment. The result of interest may be the number of different machines which break down (once or more) during the week. The sample space for this experiment then consists of the following basic outcomes:

Basic outcome #1 zero machines break down;
#2 one machine breaks down;
#3 two machines break down;
#4 three machines break down.

Note that the basic outcomes are defined such that one and only one of the basic outcomes actually occurs in the random experiment.

**Problem 1**

(a) In a marketing survey, a housewife is asked to select from a group of three different package designs (*A, B, C*) the one which she finds to be most attractive. Develop the sample space for this random experiment. √ (For answer, see page 38.)

(b) Consider the marketing survey described in (a). However, suppose that the housewife is asked to rank the three designs from the most attractive to the least attractive. Develop the sample space for this random experiment. √ (For answer, see page 38.)

B. **Univariate and Bivariate Sample Spaces.** Sample spaces in which basic outcomes refer to a single characteristic are called *univariate* sample spaces. Analogously, if the basic outcomes refer to two characteristics, the sample space is *bivariate* and if the basic outcomes refer to more than two characteristics, the sample space is *multivariate.*

Example 2 Consider the number of replacement parts of Type M43 which will be required by each of two machines during the next production period. For simplicity assume that the number of replacement parts required by a machine will not exceed two. The sample space of this random experiment contains the nine basic outcomes displayed in Table 5.1 below, where $X$ is the number of replacement parts required by machine #1 and $Y$ is the number of replacement parts required by machine #2.

**Table 5.1**

| | | Y = 0 | Y = 1 | Y = 2 |
|---|---|---|---|---|
| X | 0 | | | |
| | 1 | | | |
| | 2 | * | | |

This is a bivariate sample space because each basic outcome is represented by two characteristics, $X$ and $Y$. For instance, one basic outcome is machine #1 requiring two replacement parts ($X = 2$) and machine #2 requiring zero replacement parts ($Y = 0$). The cell in Table 5.1 representing this basic outcome has been marked by an asterisk.

**Problem 2**

Determine the number of characteristics that describe each basic outcome of the random experiment, and identify them, in:

(a) Example 1 _____

(b) Problem 1(a) _____

(c) Problem 1(b) _____ √

**C. Probability Distribution.** The assignment of probabilities to each of the basic outcomes in the sample space is called a probability distribution. Probability distributions are univariate, bivariate, or multivariate according to the sample space. The sum of the probabilities in any probability distribution is equal to one.

Example 3 Consider the random experiment of Example 2. Suppose the bivariate probability distribution for that experiment is as follows:

**Table 5.2**

| | | Y = 0 | Y = 1 | Y = 2 |
|---|---|---|---|---|
| X | 0 | .30 | .14 | .06 |
| | 1 | .18 | .10 | .02 |
| | 2 | .12 | .06 | .02 |

Reading the table, we see that the probability that $X = 2$ and $Y = 0$ is .12. This probability statement is written symbolically as $P(X = 2 \text{ and } Y = 0) = .12$. Probability statements that involve a specification on two or more characteristics are called *joint probabilities.* In some books, the word "and" in the probability statement is replaced by the set notation, "$\cap$"; for instance, $P(X = 2 \cap Y = 0)$ rather than $P(X = 2 \text{ and } Y = 0)$.

**Problem 3**

From the probability distribution in Example 3, specify the following joint probabilities.

(a) $P(X = 0 \text{ and } Y = 0) =$ _____

(b) $P(X = 1 \text{ and } Y = 0) =$ _____

(c) $P(Y = 2 \text{ and } X = 2) =$ _____ $\surd$

**D. Marginal Probabilities and Marginal Probability Distributions.** Often, when the sample space is bivariate or multivariate, we are interested in each characteristic by itself.

Example 4 For the probability distribution in Example 3, we may be interested in studying the characteristic $X$ by itself, that is, studying the number of replacement parts required by machine #1. If we sum the probabilities in each of the rows of the table in Example 3, we obtain the marginal probabilities and hence the marginal probability distribution for $X$. This distribution is:

**Table 5.3**

| $X$ | | $P(X)$ |
|---|---|---|
| 0 | .30 + .14 + .06 = | .5 |
| 1 | .18 + .10 + .02 = | .3 |
| 2 | .12 + .06 + .02 = | .2 |
| | | 1.0 |

Marginal probabilities are written symbolically, as $P(X = 0)$, $P(X = 1)$, and so on. For instance, from the marginal probability distribution for $X$ we see that $P(X = 2) = .2$.

**Problem 4**

(a) For the probability distribution in Example 3, construct the marginal probability distribution for $Y$.

(b) $P(Y = 1) =$ _____ $\surd$ (For answers, see page 40.)

**E. Conditional Probabilities and Conditional Probability Distributions.** Often, when the sample space is bivariate or multivariate, we are interested in making probability statements which are conditional on known information about the other characteristic or characteristics.

Example 5 For the probability distribution in Example 3, we may be interested in the probability distribution for $Y$, given the information that $X = 0$, that is, the probability distribution for the number of replacement parts required by machine #2 given that machine #1 will not require any. To obtain this probability distribution, each of the joint probabilities in the $X = 0$ row in the table in Example 3 is divided by the marginal probability $P(X = 0)$. These calculations are given in Table 5.4.

Answers

1. (a) The sample space would consist of the following three basic outcomes:

Basic Outcome #1. The housewife selects design *A*;
#2. The housewife selects design *B*;
#3. The housewife selects design *C*.

(b) The sample space would consist of the following six basic outcomes:

| Basic Outcome | Most Attractive | Second Choice | Least Attractive |
|---|---|---|---|
| #1 | *A* | *B* | *C* |
| #2 | *A* | *C* | *B* |
| #3 | *B* | *A* | *C* |
| #4 | *B* | *C* | *A* |
| #5 | *C* | *A* | *B* |
| #6 | *C* | *B* | *A* |

2. (a) One characteristic — number of machines which break down.
(b) One characteristic — most attractive design.
(c) Three characteristics — first, second and third choices with respect to attractiveness of design.

3. (a) $P(X = 0 \text{ and } Y = 0) = .30$.

(b) $P(X = 1 \text{ and } Y = 0) = .18$.

(c) $P(Y = 2 \text{ and } X = 2) = P(X = 2 \text{ and } Y = 2) = .02$.

Table 5.4

| $Y$ | | $P(Y \mid X = 0)$ |
|---|---|---|
| 0 | $P(Y = 0 \mid X = 0) = \frac{P(X = 0 \text{ and } Y = 0)}{P(X = 0)} = \frac{.30}{.50} =$ | .60 |
| 1 | $P(Y = 1 \mid X = 0) = \frac{P(X = 0 \text{ and } Y = 1)}{P(X = 0)} = \frac{.14}{.50} =$ | .28 |
| 2 | $P(Y = 2 \mid X = 0) = \frac{P(X = 0 \text{ and } Y = 2)}{P(X = 0)} = \frac{.06}{.50} =$ | $\frac{.12}{1.00}$ |

Probability statements of the type $P(Y = 0 \mid X = 0)$ are called conditional probabilities. $P(Y = 0 \mid X = 0)$ is read as "the probability that $Y = 0$ given that $X = 0$." The probability distribution in Table 5.4 is called a conditional probability distribution; in particular, it is the conditional probability distribution of $Y$, given that $X = 0$.

**Problem 5**

For the probability distribution in Example 3, determine the conditional probability distribution of $X$, given that $Y = 0$. $\surd$

**Problem 6**

For the probability distribution in Example 3, determine the conditional probability distribution of $Y$, given that $X = 1$. √

**F. Events.** Often, interest will center on outcomes that are broader than the basic outcomes of the sample space. An *event* consists of one or more basic outcomes of the sample space. Note that a basic outcome is also an event.

Two events defined on the same sample space are *mutually exclusive* if they do not contain common basic outcomes. Letting the symbol $E$ denote an event, the *complementary event,* "not $E$," to the event $E$, consists of all the basic outcomes in the sample space that are not contained in $E$.

Example 6 With reference to Example 1, we illustrate some events:

$E_1$ zero machines break down;
$E_2$ one machine breaks down;
$E_3$ one or more machines break down.

$E_1$ and $E_2$ are mutually exclusive events. $E_3$ is the complementary event to $E_1$.

**Problem 7**

Consider the sample space of Example 2 and the following three events:

$$A: X = 0 \qquad B: (X = 0 \text{ and } Y = 0) \qquad C: X + Y > 0.$$

(a) What basic outcomes comprise each of the events?

$A$: _____

$B$: _____

$C$: _____ √

(b) Are $A$ and $C$ mutually exclusive events? _____ Why or why not? _____

_____

(c) What is the complementary event to $B$? _____

(d) Describe event $C$ in words. _____

_____ √

**Problem 8**

Consider the sample space of Example 2 and the following two events:

$$A: (X \neq 2 \text{ and } Y \neq 0) \qquad B: X > Y.$$

Answers

4. (a)

| $Y$ | | $P(Y)$ |
|---|---|---|
| 0 | .30 + .18 + .12 = | .6 |
| 1 | .14 + .10 + .06 = | .3 |
| 2 | .06 + .02 + .02 = | .1 |
| | | 1.0 |

(b) $P(Y = 1) = .3$.

5.

| $X$ | | $P(X \mid Y = 0)$ |
|---|---|---|
| 0 | $P(X = 0 \mid Y = 0) = \frac{P(X = 0 \text{ and } Y = 0)}{P(Y = 0)} = \frac{.30}{.60} =$ | .50 |
| 1 | $P(X = 1 \mid Y = 0) = \frac{P(X = 1 \text{ and } Y = 0)}{P(Y = 0)} = \frac{.18}{.60} =$ | .30 |
| 2 | $P(X = 2 \mid Y = 0) = \frac{P(X = 2 \text{ and } Y = 0)}{P(Y = 0)} = \frac{.12}{.60} =$ | .20 / 1.00 |

**If your answer to Problem 5 was correct, you may omit Problem 6; otherwise do Problem 6.**

6.

| $Y$ | | $P(Y \mid X = 1)$ |
|---|---|---|
| 0 | $P(Y = 0 \mid X = 1) = \frac{P(Y = 0 \text{ and } X = 1)}{P(X = 1)} = \frac{.18}{.30} =$ | .60 |
| 1 | $P(Y = 1 \mid X = 1) = \frac{P(Y = 1 \text{ and } X = 1)}{P(X = 1)} = \frac{.10}{.30} =$ | .33 |
| 2 | $P(Y = 2 \mid X = 1) = \frac{P(Y = 2 \text{ and } X = 1)}{P(X = 1)} = \frac{.02}{.30} =$ | .07 / 1.00 |

7. (a) $A$ consists of ($X = 0$ and $Y = 0$), ($X = 0$ and $Y = 1$), ($X = 0$ and $Y = 2$).
$B$ consists of one basic outcome, ($X = 0$ and $Y = 0$).
$C$ consists of all basic outcomes except ($X = 0$ and $Y = 0$).
(b) No, because $A$ and $C$ have the basic outcomes ($X = 0$ and $Y = 1$) and ($X = 0$ and $Y = 2$) in common.
(c) $C$ is the complementary event to $B$, i.e., $C$ = not $B$.
(d) $C$ is the event that the total number of replacement parts required by both machines exceeds zero.

**If you didn't find Problem 7 difficult, good work, you may omit Problem 8; otherwise try Problem 8.**

(a) What are the basic outcomes that comprise each of these events?

$A$: _____

$B$: _____ √

(b) Are $A$ and $B$ mutually exclusive events? _____

(c) What is the complementary event to $A$? _____

(d) Describe event $B$ in words. _____ √

**G. Basic Probability Theorems.** The following probability theorems apply to events:

I. The probability of an event is the sum of the probabilities of the basic outcomes constituting the event.
II. $P(\text{not } E) = 1 - P(E)$.
III. If $E$ and $F$ are mutually exclusive events, then $P(E \text{ and } F) = 0$.
IV. (Addition Theorem) $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.
The "or" of the addition theorem is the inclusive "or". Thus, ($E$ or $F$) means either $E$, or $F$, or both $E$ and $F$.
In some books, the set notation "∪" is used in place of "or", that is, $P(E \cup F)$ rather than $P(E \text{ or } F)$.
V. (Multiplication Theorem) $P(E \text{ and } F) = P(E)P(F \mid E) = P(F)P(E \mid F)$.

**Problem 9**

Consider the probability distribution in Example 3 and the following events. $A$: $X = 0$, $B$: $Y \neq 0$, $C$: $X + Y > 0$.

(a) From the probability distribution, obtain the following probabilities:
$P(A)$ = _____

$P(B)$ = _____

$P(C)$ = _____

$P(A \text{ and } B)$ = _____ √

(b) Using the addition or multiplication theorem, calculate the following probabilities:
$P(A \text{ or } B)$ = _____

$P(A \mid B)$ = _____

$P(B \mid A)$ = _____

$P(A \text{ and } B)$ = _____ √

(c) Using the theorem for complementary events, calculate $P(C)$ _____
_____ √

**Problem 10**

Consider the probability distribution in Example 3 and the following events:

$$A: X > Y; \quad B: X = Y.$$

(a) From the probability distribution, obtain the following probabilities:
$P(A)$ = _____

$P(B)$ = _____

$P(A \text{ and } B)$ = _____

$P(\text{not } B)$ = _____

$P(A \text{ and not } B)$ = _____ √

(For answers, see page 44.)

**Answers**

8. (a) $A$ consists of ($X = 0$ and $Y = 1$), ($X = 0$ and $Y = 2$), ($X = 1$ and $Y = 1$), ($X = 1$ and $Y = 2$). $B$ consists of ($X = 1$ and $Y = 0$), ($X = 2$ and $Y = 0$), ($X = 2$ and $Y = 1$).
   (b) Yes. $A$ and $B$ have no basic outcomes in common.
   (c) not $A$ consists of ($X = 0$ and $Y = 0$), ($X = 1$ and $Y = 0$), ($X = 2$ and $Y = 0$), ($X = 2$ and $Y = 1$), ($X = 2$ and $Y = 2$).
   (d) $B$ is the event that machine #1 requires more replacement parts than machine #2.

9. (a) $P(A) = P(X = 0) = .30 + .14 + .06 = .50$;
   $P(B) = P(Y \neq 0) = .14 + .10 + .06 + .06 + .02 + .02 = .40$;
   $P(C) = P(X + Y > 0) = .18 + .12 + .14 + .10 + .06 + .06 + .02 + .02 = .70$;
   $P(A \text{ and } B) = P(X = 0 \text{ and } Y \neq 0) = .14 + .06 = .20$.

   (b) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = .50 + .40 - .20 = .70$;
   $P(A \mid B) = P(A \text{ and } B)/P(B) = .20/.40 = .50$;
   $P(B \mid A) = P(A \text{ and } B)/P(A) = .20/.50 = .40$;
   $P(A \text{ and } B) = P(A)P(B \mid A) = (.50)(.40) = .20$, or $P(B)P(A \mid B) = (.40)(.50) = .20$.

   (c) $P(C) = 1 - P(\text{not } C) = 1 - P(X + Y = 0) = 1 - .30 = .70$.

**If you didn't find Problem 9 difficult, good work, you may omit Problem 10; otherwise try Problem 10.**

(b) Using the addition or multiplication theorem, calculate the following probabilities:

$P(A \text{ or } B) =$ _____

$P(A \mid \text{not } B) =$ _____ √

**H. Statistical Independence.**

Example 7 Two *events*, $E$ and $F$, defined on the same sample space, are statistically independent if their joint probability is equal to the product of the two marginal probabilities, that is, if $P(E \text{ and } F) = P(E)P(F)$. For instance, in the probability distribution in Example 3, the two events, $E$: $X = 0$ and $F$: $Y = 0$, are statistically independent because $P(X = 0 \text{ and } Y = 0) = .30$ and $P(X = 0)P(Y = 0) = (.60)(.50) = .30$.

**Problem 11**

Consider the probability distribution in Example 3 and the two events, $A$: $X = 1$ and $B$: $Y = 1$. Are $A$ and $B$ statistically independent? _____ √

Example 8 Two *variables* (for instance, $X$ and $Y$ in Example 3) are statistically independent if *all* joint probabilities equal the product of their corresponding marginal probabilities. For $X$ and $Y$ in Example 3 to be statistically independent, the above definition requires that $P(X \text{ and } Y) = P(X)P(Y)$ for all possible pairs of values which $X$ and $Y$ may take on.

An inspection of the probability distribution in Example 3 indicates that $X$ and $Y$ are not statistically independent. One violation of the above requirement is $P(X = 1 \text{ and } Y = 1) = .10$, whereas $P(X = 1)\,P(Y = 1) = (.30)(.30) = .09$.

**Problem 12**

A shipment of spare parts is being inspected. Each part in the shipment has been produced on one of two machines, $M_1$ or $M_2$, and each part is graded as being defective ($D$) or acceptable ($A$). The probability distribution in Table 5.5 applies to a part selected at random from the shipment:

**Table 5.5**

| | | Grade | | |
|---|---|---|---|---|
| | | $D$ | $A$ | Total |
| Machine | $M_1$ | .10 | .50 | .60 |
| | $M_2$ | .05 | .35 | .40 |
| | Total | .15 | .85 | 1.00 |

(a) Are the two variables, grade of a part and machine, statistically independent?____Why or why not?

_____ √

(b) Keeping the marginal probabilities the same as in the above probability distribution, what would the joint probabilities have to be in order for grade and machine to be statistically independent variables? √

**I. Bayes' Theorem.** This theorem represents an application of conditional probabilities. It permits us to revise prior probabilities on the basis of subsequent information.

Example 9 Consider the joint probability distribution in Problem 12. Prior to the random selection of a part from the shipment, the marginal probability is .60 that the part is produced on machine $M_1$ and .40 that it is produced on machine $M_2$. Suppose the selected part is graded and found to be defective. This information may be used to revise the above probabilities. The revised probabilities which are sought are the conditional probabilities $P(M_1 \mid D)$ and $P(M_2 \mid D)$. The multiplication theorem, when applied to the probabilities of Problem 12, gives:

$$P(M_1 \mid D) = \frac{P(M_1 \text{ and } D)}{P(D)} = \frac{.10}{.15} = .67;$$

$$P(M_2 \mid D) = \frac{P(M_2 \text{ and } D)}{P(D)} = \frac{.05}{.15} = .33.$$

In the terminology of Bayes' Theorem, $P(M_1)$ and $P(M_2)$ are called *prior probabilities* and $P(M_1 \mid D)$ and $P(M_2 \mid D)$ are called *posterior probabilities.* We can summarize these probabilities for our example as follows:

**Table 5.6**

| Machine | Prior Probability | Posterior Probability Given that the Selected Part is Defective |
|---|---|---|
| $M_1$ | .60 | .67 |
| $M_2$ | .40 | .33 |
| | 1.00 | 1.00 |

**Answers**

10. (a) $P(A) = P(X > Y) = .18 + .12 + .06 = .36$;
$P(B) = P(X = Y) = .30 + .10 + .02 = .42$;
$P(A \text{ and } B) = P(X > Y \text{ and } X = Y) = 0$ because $A$ and $B$ are mutually exclusive events;
$P(\text{not } B) = P(X \neq Y) = .18 + .12 + .06 + .14 + .06 + .02 = .58$;
$P(A \text{ and not } B) = P(X > Y \text{ and } X \neq Y) = .18 + .12 + .06 = .36$.

(b) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = .36 + .42 - 0 = .78$;
$P(A \mid \text{not } B) = P(A \text{ and not } B)/P(\text{not } B) = .36/.58 = .62$.

11. $P(A) = P(X = 1) = .30$, $P(B) = P(Y = 1) = .30$;
$P(A \text{ and } B) = P(X = 1 \text{ and } Y = 1) = .10$;
$P(A \text{ and } B) = .10 \neq P(A)P(B) = (.30)(.30) = .09$. Thus, events $A$ and $B$ are not statistically independent.

12. (a) **No, because all joint probabilities are not equal to corresponding products of marginal probabilities; for instance:**

$$P(M_1 \text{ and } D) = .10 \neq P(M_1)P(D) = (.60)(.15) = .09$$

(b)

| | $D$ | $A$ | Total |
|---|---|---|---|
| $M_1$ | $(.15)(.60) = .09$ | $(.85)(.60) = .51$ | .60 |
| $M_2$ | $(.15)(.40) = .06$ | $(.85)(.40) = .34$ | .40 |
| Total | .15 | .85 | 1.00 |

To use Bayes' Theorem, one need not know the complete joint probability distribution. The posterior probabilities $P(M_i \mid D)$ can be obtained simply from the prior probabilities $P(M_i)$ and the conditional probabilities $P(D \mid M_i)$, as follows:

$$P(M_i \mid D) = \frac{P(M_i \text{ and } D)}{P(D)} = \frac{P(M_i)\ P(D \mid M_i)}{P(M_1)P(D \mid M_1) + P(M_2)P(D \mid M_2)}$$

A similar expression can be obtained for the posterior probability $P(M_i \mid A)$.

**Problem 13**

Consider the probability distribution in Problem 12. Obtain the posterior probability distribution for the machine which produced the part, given that the part selected at random is acceptable. √

## REVIEW PROBLEMS

1. A pool of typists in the Central Transcribing Office serves four departments of the firm ($A$, $B$, $C$, $D$). Typing requests are sent to the pool marked "Rush" or "Routine." A typing request is selected at random from those which are sent to the pool. Develop the bivariate sample space of this random experiment. √

2. A department store undertook a study of customer purchases, according to the dollar size of the customer's purchase and whether or not the purchase is charged. The symbols $B_1$, $B_2$, and $B_3$ denote the size of purchase classes and $A_1$ and $A_2$ denote the mode of payment classes. The probability distribution is as follows:

**Table 5.7**

| | | Size of Purchase | | |
|---|---|---|---|---|
| | | $B_1$ | $B_2$ | $B_3$ |
| | | Less than \$5 | \$5–\$20 | Over \$20 |
| Mode of Payment | Cash $A_1$ | .25 | .15 | .10 |
| | Charge $A_2$ | .15 | .20 | .15 |

(a) Construct the marginal probability distribution for the mode of payment. √ (For answer, see page 48.)

(b) Construct the conditional probability distribution for the size of purchase given that the customer is making a cash purchase. √ (For answer, see page 48.)

(c) What is the probability that a customer will make a purchase of \$20 or less (given no information about the mode of payment)? _____

(d) A customer has a \$23 purchase. What is the posterior probability distribution for the mode of payment of that customer? √ (For answers, see page 48.)

Answers

13.

| Machine | | Posterior Probability Given that the Selected Part is Acceptable |
|---|---|---|
| $M_1$ | $P(M_1 \mid A) = P(M_1 \text{ and } A) / P(A) = .50/.85 =$ | .59 |
| $M_2$ | $P(M_2 \mid A) = P(M_2 \text{ and } A) / P(A) = .35/.85 =$ | .41 |
| | | 1.00 |

## ANSWERS TO REVIEW PROBLEMS

1.

| | | Department | | | |
|---|---|---|---|---|---|
| | | A | B | C | D |
| Priority | Rush | | | | |
| | Routine | | | | |

(e) Determine the following probabilities:

$P(A_2 \text{ or } B_3) =$ _____

$P(A_1 \text{ and not } B_3) =$ _____

(f) Are the variables, size of purchase and mode of payment, statistically independent? ____Why or why not? _____ √

3. In any given week a salesman can either sell no machine or one machine. The probability that Salesman A sells one machine is .20, and for Salesman B this probability is .30. Assuming that the outcomes for the two salesmen are statistically independent, what is the probability that their total sales for the week will not exceed one machine? _____ √

## ANSWERS TO REVIEW PROBLEMS

2. (a)

| Mode of Payment | | Probability |
|---|---|---|
| Cash | .25 + .15 + .10 = | .50 |
| Charge | .15 + .20 + .15 = | .50 |
| | | 1.00 |

(b)

| Size of Purchase | Probability, Given a Cash Purchase |
|---|---|
| Less than $5 | .25/.50 = .50 |
| $5–$20 | .15/.50 = .30 |
| Over $20 | .10/.50 = .20 |
| | 1.00 |

(c) $P(\$20 \text{ or less}) = .25 + .15 + .15 + .20 = .75.$

(d)

| Mode of Payment | | Posterior Probability, Given That the Purchase is for $23 |
|---|---|---|
| Cash | $P(A_1 \mid B_3) = .10/.25 =$ | .40 |
| Charge | $P(A_2 \mid B_3) = .15/.25 =$ | .60 |
| | | 1.00 |

(e) $P(A_2 \text{ or } B_3) = P(A_2) + P(B_3) - P(A_2 \text{ and } B_3) = .50 + .25 - .15 = .60$
$P(A_1 \text{ and not } B_3) = .25 + .15 = .40$

(f) No, because all joint probabilities are not equal to corresponding products of marginal probabilities; for instance:

$$P(A_1 \text{ and } B_1) = .25 \neq P(A_1)P(B_1) = (.50)(.40) = .20.$$

3.

| | | Sales by B | | |
|---|---|---|---|---|
| | | None | One | Total |
| Sales by A | None | .56 | .24 | .80 |
| | One | .14 | .06 | .20 |
| | Total | .70 | .30 | 1.00 |

The assumption of statistical independence permits us to obtain the joint probabilities as the products of corresponding marginal probabilities; for instance, (.80) (.70) = .56.

$P(\text{total sales} \leqslant 1) = .56 + .24 + .14 = .94.$

# 6

# Random Variables

A random variable is a variable whose value is determined by a random experiment.

**A. Probability Distribution of Random Variable.**

Example 1 Consider a random experiment involving the performances of three machines during a one-week production period. The number of machines that fail during the week is the random variable of interest. Let $x$ denote this random variable. Assume that $x$ has the following probability distribution:

| $X$ | $P(X)$ |
|---|---|
| 0 | .70 |
| 1 | .15 |
| 2 | .10 |
| 3 | .05 |
| | 1.00 |

Note that small $x$ here denotes the random variable and capital $X$ the actual outcome or value.

**Problem 1**

(a) Given the probability distribution for $x$ in Example 1, construct below the probability distribution of the random variable $y$ representing the number of machines that do *not* fail during the week.

a

(b) Assume a machine failure costs the firm \$700. From the probability distribution of $x$ in Example 1, construct the probability distribution of the random variable $z$ representing the cost of machine failures.

(For answers, see page 52.) √

b

**B. Expectation of Random Variable.** The *expected value* of a random variable $x$ is denoted $E(x)$. When the random variable is discrete, $E(x)$ is defined as follows:

$$E(x) = \Sigma\, XP(X).$$

$E(x)$ is the mean of the probability distribution of $x$. Thus $E(x)$ corresponds to $\mu$, the population mean.

Example 2 Consider the probability distribution in Example 1. The expected number of failures is obtained as follows:

$$E(x) = 0(.70) + 1(.15) + 2(.10) + 3(.05) = .5.$$

**Problem 2**

The number of television sets that are serviced each hour (i.e., completed) at a television repair center has the following probability distribution:

| $X$ | $P(X)$ |
|---|---|
| 0 | .5 |
| 1 | .3 |
| 2 | .2 |
| | 1.0 |

(a) What is the expected number of sets serviced each hour? _____
_____ √

(b) Can the expected number of sets be interpreted as a weighted average? _____ Explain.
_____

(c) Your answer in (a) is a fraction, yet the number of television sets serviced in any hour must be a whole number. Is this an inconsistency? _____
_____
_____ √

Example 3 Frequently one is interested in the expected value of a random variable $w$ which is a function of a random variable $x$, and $E(x)$ is already known. If $w = kx + c$, where $k$ and $c$ are specified constants, the following theorem can be used:

$$E(w) = kE(x) + c.$$

For instance, in (a) of Problem 1, the number of machines that do not fail ($y$) is related to the number of machines that do fail ($x$) by the formula $y = 3 - x$. In Example 2 it was shown that $E(x) = .5$. Using the above formula, it follows that $E(y) = 3 - E(x) = 2.5$. Thus, the expected number of machines that do not fail is 2.5.

**Problem 3**

Refer to Problem 2. If the service center charges \$8 for every television it repairs, what are the expected receipts per hour? _____ √

**C. Variance of Random Variable.** The variance of a discrete probability distribution is obtained using the following formula:

$$\sigma^2 = E(x - \mu)^2 = \Sigma(X - \mu)^2\, P(X).$$

The variance of a probability distribution corresponds to the variance of a population. Hence, the same symbol $\sigma^2$ is used for both. The *standard deviation* of a probability distribution is denoted by $\sigma$, and equals the square root of $\sigma^2$.

Example 4 Consider the probability distribution of $x$ in Example 1. It was shown in Example 2 that $E(x) = \mu = .5$. The variance of this probability distribution is:

$$\sigma^2 = (0 - .5)^2(.70) + (1 - .5)^2(.15) + (2 - .5)^2(.10) + (3 - .5)^2(.05)$$
$$= .1750 + .0375 + .2250 + .3125 = .75.$$

The standard deviation equals $\sigma = \sqrt{.75} = .87$.

**Problem 4**

Determine the variance and standard deviation of the probability distribution in Problem 2.
$\sigma^2 =$ _____ ;
$\sigma =$ _____ .√

Example 5 If one is interested in the variance of a random variable $w$ where $w = kx + c$, and $\sigma^2$ for $x$ is already known, the following theorem may be used (here $k$ and $c$ are specified constants):

$$\text{variance of } w = k^2\sigma^2.$$

Note that adding a constant $c$ to a variable does not change its variance.

To illustrate the use of the formula, in (a) of Problem 1, $y = 3 - x$ where $y$ is the number of machines that do not fail and $x$ the number of machines that do fail. In Example 4, the variance of $x$ was found to be .75. Using the above formula, it follows that:

$$\text{variance of } y = (-1)^2(.75) = .75.$$

Hence, the variances of $x$ and $y$ are equal in this example.

**Problem 5**

Refer to Problem 3. What are the variance and standard deviation of the receipts per hour? (*Hint*: Use the results of Problem 4.) _____ √

**D. Standardized Variable.** If $x$ is a random variable with mean $\mu$ and standard deviation $\sigma$, then a standardized random variable $z$ is defined:

$$z = \frac{x - \mu}{\sigma}.$$

The mean and standard deviation of $z$ are zero and one respectively.

Example 6 To show that the mean of $z$ is zero, note that $z$ may be written:

$$z = (x - \mu)/\sigma = (1/\sigma)x - (\mu/\sigma).$$

Hence the expected value of $z$ is:

$$E(z) = (1/\sigma)E(x) - (\mu/\sigma) = (1/\sigma)\mu - (\mu/\sigma) = 0.$$

**Problem 6**

Show that the standard deviation of $z$ is one. _____ √

(For answer, see page 54.)

**Answers**

1. (a)

| Number of Machines That Do Not Fail $Y$ | Probability |
|---|---|
| 0 | .05 |
| 1 | .10 |
| 2 | .15 |
| 3 | .70 |
| | 1.00 |

The number of machines that do not fail ($y$) is obtained by subtracting the number that do fail ($x$) from three, that is, $y = 3 - x$.

(b)

| Cost of Failures $Z$ | Probability |
|---|---|
| 0 | .70 |
| 700 | .15 |
| 1400 | .10 |
| 2100 | .05 |
| | 1.00 |

Since each failure costs \$700, the total cost is obtained by multiplying 700 times $x$, that is, $z = 700x$.

2. (a) $E(x) = 0(.5) + 1(.3) + 2(.2) = .7$.
   (b) Yes. The probabilities are the weights and the outcomes $X$ are the items to be averaged.
   (c) **No. The expected number of sets serviced each hour is a weighted average of whole numbers as shown in (a) above. This weighted average need not correspond to the actual number of sets serviced in any hour and, in general, it will not be a whole number.**

3. Let $w$ = receipts. Thus $w = 8x$ and $E(w) = 8\,E(x)$. In Problem 2 we determined that $E(x) = .7$. This gives $E(w) = 8(.7) = \$5.60$ per hour.

4. $\sigma^2 = (0 - .7)^2(.5) + (1 - .7)^2(.3) + (2 - .7)^2(.2) = .61$; $\sigma = \sqrt{.61} = .78$.

5. Let $w$ = receipts. Thus $w = 8x$. From Problem 4, we found $\sigma^2$ for $x$ equals .61. This gives:

   variance of $w = (8)^2(.61) = 39.04$;

   standard deviation of $w = \sqrt{39.04} = \$6.24$.

**E. Covariance of Two Random Variables.** Covariance is a measure of the statistical dependence of two random variables. It describes the extent to which they vary together. For two random variables, $x$ and $y$, the covariance of $x$ and $y$ is denoted by Cov $(X, Y)$ and is obtained in the case of discrete variables from the following formula:

$$\text{Cov}\,(X, Y) = E(x - \mu_X)(y - \mu_Y) = \Sigma(X - \mu_X)(Y - \mu_Y)P(X \text{ and } Y).$$

Here $\mu_X$ and $\mu_Y$ are the means of $X$ and $Y$ respectively. $P(X$ and $Y)$ is the joint probability of $X$ and $Y$. The summation in the formula applies to all possible pairs of $X$ and $Y$ values. In some books, the symbol $\sigma_{XY}$ is used instead of Cov $(X, Y)$.

If two random variables are statistically independent, their covariance is zero. If they are positively related, their covariance is positive; if they are negatively related, their covariance is negative.

**Example 7** Consider the bivariate probability distribution of Example 3, Section 5:

| | | $Y$ | | |
|---|---|---|---|---|
| | | 0 | 1 | 2 |
| $X$ | 0 | .30 | .14 | .06 |
| | 1 | .18 | .10 | .02 |
| | 2 | .12 | .06 | .02 |

Here $x$ and $y$ represent the number of replacement parts required by machine #1 and machine #2 respectively.

For the above probability distribution, the marginal probability distributions of $x$ and $y$ are:

| $X$ | $P(X)$ | $Y$ | $P(Y)$ |
|---|---|---|---|
| 0 | .5 | 0 | .6 |
| 1 | .3 | 1 | .3 |
| 2 | .2 | 2 | .1 |
| | 1.0 | | 1.0 |

Calculating the means of the above marginal probability distributions gives $\mu_X = .7$ and $\mu_Y = .5$. The covariance of $x$ and $y$ is based on the pairs of $X, Y$ values (0,0), (0,1), (0,2), (1,0), etc. We have:

$$\begin{aligned}\text{Cov}(X,Y) &= (0-.7)(0-.5)(.30) + (0-.7)(1-.5)(.14) + (0-.7)(2-.5)(.06) \\ &\quad + (1-.7)(0-.5)(.18) + (1-.7)(1-.5)(.10) + (1-.7)(2-.5)(.02) \\ &\quad + (2-.7)(0-.5)(.12) + (2-.7)(1-.5)(.06) + (2-.7)(2-.5)(.02) \\ &= .105 - .049 - .063 - .027 + .015 + .009 - .078 + .039 + .039 \\ &= -.01.\end{aligned}$$

**Problem 7**

A firm has undertaken a capital investment with an economic life of two years. The following probability distribution describes the profit prospects (in $ millions) of the venture in each of the two years:

| | | $Y$ = profit in second year | |
|---|---|---|---|
| | | 0 | +1 |
| $X$ = profit in first year | −1 | .3 | .1 |
| | 0 | .1 | .1 |
| | +1 | .1 | .3 |

(a) For the above probability distribution, $\mu_X = 0$ and $\mu_Y = .5$. Calculate Cov $(X,Y)$. _____ √

(For answer, see page 56.)

(b) From the sign of Cov $(X,Y)$, how are first and second year profits related? _____ √

(For answer, see page 56.)

**F. Correlation Coefficient.** The correlation coefficient $\rho$ (Greek rho) is a measure of the relative statistical dependence of two random variables. It is calculated using the following formula:

**Answers**

6. In Example 6, we saw $z = (x - \mu)/\sigma = (1/\sigma)x - (\mu/\sigma)$. Thus, variance of $z = (1/\sigma)^2$ (variance of $x$) = $(1/\sigma)^2\sigma^2 = 1$, and standard deviation is $\sqrt{1} = 1$.

$$\rho = \frac{\text{Cov}\,(X, Y)}{\sigma_X \sigma_Y}.$$

Here $\sigma_X$ and $\sigma_Y$ are the standard deviations of the random variables $x$ and $y$ respectively.

It can be shown that $\rho$ may take on values in the interval $-1 \leqslant \rho \leqslant +1$; $\rho = -1$ indicates that $x$ and $y$ exhibit perfect negative correlation, $\rho = +1$ indicates perfect positive correlation, and $\rho = 0$ indicates that $x$ and $y$ are not correlated.

**Problem 8**

(a) What is the relation between the sign of Cov $(X,Y)$ and $\rho$? _____

_____

(b) If two variables are statistically independent, then $\rho =$ ____? Why? _____

_____ √

Example 8 For the probability distribution in Example 7, $\sigma_X$ and $\sigma_Y$ may be calculated from the marginal probability distributions of $x$ and $y$; they are $\sigma_X = .78$ and $\sigma_Y = .67$. In Example 7, we saw that Cov $(X,Y) = -.01$. Thus:

$$\rho = \frac{-.01}{(.78)(.67)} = -.02.$$

**Problem 9**

Refer to Problem 7.

(a) For $x$ and $y$ in that problem, $\sigma_X = .89$ and $\sigma_Y = .50$. Calculate $\rho$. _____

_____

Are $x$ and $y$ correlated positively or negatively? _____ √

(b) The units of $x$ and $y$ are $ millions. Give the units of the following measures:

$\sigma_X$ _____ ;

$\sigma_Y^2$ _____ ;

Cov $(X,Y)$ _____ ;

$\rho$ _____ √

**G. Mean and Variance of the Sum and Difference of Two Statistically Independent Random Variables.** One is often interested in a random variable that is the sum or difference of two statistically independent random variables. Consider $z = x + y$ and $w = x - y$. If $\mu_X$ and $\mu_Y$ are known, then the following two formulas can be utilized to obtain $\mu_Z$ and $\mu_W$:

$$\mu_Z = \mu_X + \mu_Y \quad \text{and} \quad \mu_W = \mu_X - \mu_Y.$$

The following formula gives $\sigma_Z^2$ and $\sigma_W^2$:

$$\sigma_Z^2 = \sigma_W^2 = \sigma_X^2 + \sigma_Y^2.$$

Note that the variances of $x + y$ and $x - y$ are equal when $x$ and $y$ are statistically independent.

Example 9 The manufacture of an item involves two activities, cutting and polishing. Suppose that the lengths of time (in minutes) required to perform each of the activities are statistically independent random variables that have the following characteristics:

| Activity | Mean | Standard Deviation |
|---|---|---|
| Cutting | 8 | 3 |
| Polishing | 5 | 4 |

The mean and standard deviation of the total time required to perform both activities are:

$$\text{mean total time} = 8 + 5 = 13;$$
$$\text{variance of total time} = (3)^2 + (4)^2 = 25;$$
$$\text{standard deviation of total time} = \sqrt{25} = 5.$$

**Problem 10**

Consider the problem described in Example 9. What are the mean and standard deviation of the difference between the time required for cutting and the time required for polishing? _____ _____ √

**H. Mean and Variance of the Sum of Several Statistically Independent Random Variables.** If $x_1, x_2, \cdots, x_n$ are $n$ statistically independent random variables with means $\mu_{X_1}, \mu_{X_2}, \cdots, \mu_{X_n}$ and variances $\sigma^2_{X_1}, \sigma^2_{X_2}, \cdots, \sigma^2_{X_n}$, then the mean and variance of the sum $w = x_1 + x_2 + \cdots + x_n$ are given by the following formulas:

$$\mu_W = \mu_{X_1} + \mu_{X_2} + \cdots + \mu_{X_n},$$
$$\sigma^2_W = \sigma^2_{X_1} + \sigma^2_{X_2} + \cdots + \sigma^2_{X_n}.$$

**Problem 11**

A firm has four identical warehouses with the same probability distribution of loss by fire for each warehouse. The mean and standard deviation of this probability distribution (in $ thousands) are 50 and 25 respectively.

(a) If fire losses in the four warehouses are statistically independent, find the mean and the standard deviation of the total fire loss for the four warehouses.
_____ _____ √

(b) How great is the mean total fire loss in four warehouses, relative to that for one warehouse? _____ _____

(c) How great is the standard deviation of the total fire loss in four warehouses, relative to that for one warehouse? _____

(d) What does this imply about relative variation in the two distributions? Can the total fire loss in four warehouses be predicted relatively more precisely than it can in one warehouse? _____ _____ √

**I. Chebyshev Inequality.** The Chebyshev theorem states that for *any* probability distribution, the probability of an item falling beyond $k$ standard deviations from the mean is at most $1/k^2$.

Example 10 In Example 9, the total time for cutting and polishing had a mean of 13 minutes and a standard deviation of 5 minutes. The Chebyshev theorem tells us, for example, that the probability the total time will be less than 3 minutes or greater than 23 minutes is at most $1/2^2 = 1/4$. This result follows from the fact that 3 and 23 are each 2 standard deviations from the mean of 13, so $k = 2$ here.

Answers

7. (a) $\text{Cov}(X,Y) = (-1-0)(0-.5)(.3) + (-1-0)(+1-.5)(.1) + (0-0)(0-.5)(.1) + (0-0)(+1-.5)(.1)$
$+ (+1-0)(0-.5)(.1) + (+1-0)(+1-.5)(.3) = +.2.$

(b) The sign of Cov $(X,Y)$ is +, thus first and second year profits are positively related.

8. (a) $\rho = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$. Since $\sigma_X$ and $\sigma_Y$ are always positive, $\rho$ and Cov $(X,Y)$ will always have the same sign.

(b) $\rho = 0$. Because Cov $(X,Y) = 0$ when $x$ and $y$ are statistically independent.

9. (a) $\rho = \dfrac{+.2}{(.89)(.50)} = +.45$. Hence $x$ and $y$ are positively related.

(b) $\sigma_X$ : \$ millions; $\sigma_Y^2$ : (\$ millions)$^2$; Cov $(X,Y)$ : (\$ millions)$^2$; $\rho$ : unitless value since

$$\text{units of } \rho = \frac{(\$ \text{ millions})^2}{(\$ \text{ millions})(\$ \text{ millions})} = \text{unitless value.}$$

10. Mean of difference = $8 - 5 = 3$ minutes.
Variance of difference = $(3)^2 + (4)^2 = 25$.
Standard deviation of difference = $\sqrt{25} = 5$ minutes.

11. (a) Mean of the total fire loss = $50 + 50 + 50 + 50 = \$200$ thousand.
Variance of the total fire loss = $(25)^2 + (25)^2 + (25)^2 + (25)^2 = 2500$.
Standard deviation of the total fire loss = $\sqrt{2500} = \$50$ thousand.

(b) $\dfrac{\text{Mean total fire loss in four warehouses}}{\text{Mean fire loss in one warehouse}} = \dfrac{200}{50} = 4.$

(c) $\dfrac{\text{Standard deviation of the total fire loss in four warehouses}}{\text{Standard deviation of the fire loss in one warehouse}} = \dfrac{50}{25} = 2.$

(d) Coefficient of variation = $\dfrac{50}{200} \times 100 = 25\%$ for four warehouses.

Coefficient of variation = $\dfrac{25}{50} \times 100 = 50\%$ for one warehouse.

The coefficient of variation, and hence the relative variability, is smaller for the distribution involving four warehouses. Thus, the total loss for four warehouses can be predicted relatively more precisely.

**Problem 12**

Refer to Problem 11. At most, what is the probability that the total fire loss for all four warehouses will differ from the expected loss by more than \$120 thousand?_____

_____ √

## REVIEW PROBLEMS

1. The probability that a farmer's wheat crop will be a specified grade is:

| Grade | Probability |
|---|---|
| No. 1 | .8 |
| No. 2 | .2 |
| | 1.0 |

The per bushel prices for the two grades are: No. 1: \$1.90 and No. 2: \$1.60.

(a) What is the farmer's expected revenue per bushel? _____

(b) What is the standard deviation of the revenue per bushel? _____ √

2. A food company has advertised a new product twice on television. A survey of shoppers indicates that the following bivariate probability distribution describes the relationship between the number of times the shopper recalls seeing the advertisement and the number of times the shopper has purchased a package of the product (maximum is one purchase):

| | | | Number of Times Advertisement Seen $Y$ | | |
|---|---|---|---|---|---|
| | | | 0 | 1 | 2 |
| No Purchase | $X$ | 0 | .39 | .15 | .36 |
| Purchase | | 1 | .01 | .05 | .04 |

(a) Calculate Cov $(X,Y)$ and $\rho$ for the above bivariate probability distribution. _____ _____

(b) Can $\rho$ be zero if Cov $(X,Y)$ is not zero? _____

_____ √

3. A company received three shipments of a product. The mean and standard deviation of the number of defective units in each of the shipments are given below:

| Shipment | Mean | Standard deviation |
|---|---|---|
| A | 6 | 2 |
| B | 12 | 4 |
| C | 15 | 4 |

(a) If the number of defective units in each shipment are statistically independent of one another, determine the mean and standard deviation of the total number of defective units in all three shipments.

_____ _____

(b) A one hundred per cent inspection of shipment A reveals eight defective units. What are the mean and standard deviation of the total number of defective units in the remaining two shipments?

_____ _____

(c) Determine an upper bound for the probability that the number of defective units in shipment B will differ from the expected number by more than 12 units. _____ √

Answers

12. $120 thousand is 120/50 = 2.4 standard deviations. Hence, $k = 2.4$ and $1/k^2 = 1/(2.4)^2 = .17$.

## ANSWERS TO REVIEW PROBLEMS

**1. (a)** $1.90(.8) + 1.60(.2) = \$1.84$.

**(b)** $(1.90 - 1.84)^2(.8) + (1.60 - 1.84)^2(.2) = .0144$; standard deviation $= \sqrt{.0144} = \$0.12$.

**2. (a)**

| $X$ | $P(X)$ | | $Y$ | $P(Y)$ | |
|---|---|---|---|---|---|
| 0 | .9 | $\mu_X = .1$ | 0 | .4 | $\mu_Y = 1.0$ |
| 1 | .1 | $\sigma_X^2 = .09$ | 1 | .2 | $\sigma_Y^2 = .8$ |
| | 1.0 | $\sigma_X = .30$ | 2 | .4 | $\sigma_Y = .89$ |
| | | | | 1.0 | |

$$\text{Cov}(X,Y) = (0 - .1)(0 - 1)(.39) + (0 - .1)(1 - 1)(.15) + (0 - .1)(2 - 1)(.36) + (1 - .1)(0 - 1)(.01) + (1 - .1)(1 - 1)(.05) + (1 - .1)(2 - 1)(.04) = +.03;$$

$$\rho = \frac{.03}{(.30)(.89)} = +.11.$$

(b) Since $\rho = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$, $\rho$ cannot be zero if $\text{Cov}(X,Y)$ is not zero.

**3. (a)** Mean = 6 + 12 + 15 = 33.
Variance = $(2)^2 + (4)^2 + (4)^2 = 36$.
Standard deviation = $\sqrt{36} = 6$.

**(b)** Because statistical independence has been assumed, the result of the complete inspection of shipment A provides us with no new information about the number of defective units in the remaining two shipments. Hence, for shipments B and C we have:

mean = 12 + 15 = 27;
variance = $(4)^2 + (4)^2 = 32$;
standard deviation = $\sqrt{32} = 5.66$.

**(c)** 12 units is 12/4 = 3 standard deviations. Hence, using the Chebyshev inequality, $k = 3$ and $1/k^2 = 1/9$.

# 7

# Common Probability Distributions

It has been found that a limited number of probability distributions can be employed successfully in a wide range of problems.

**A. Binomial Probability Distribution.** The binomial probability distribution is described by the following mathematical formula:

$$P(X) = \binom{n}{X} p^X (1-p)^{n-X} \text{ for } X = 0, 1, 2, \cdots, n,$$

where $p$ is a parameter, and $\binom{n}{X}$ is the symbol for the number of combinations of $n$ things taken $X$ at a time:

$$\binom{n}{X} = \frac{n!}{X!(n-X)!}.$$

Remember that $A! = A(A-1)(A-2)\cdots(2)(1)$, and $0! = 1$.

The binomial probability distribution has a mean $\mu = np$ and a standard deviation $\sigma = \sqrt{np(1-p)}$.

**Example 1** The probability that a person will respond to a mail questionnaire is .4. Assume that mail questionnaires have been sent to two persons, whose decisions on responding are independent. The formula for the binomial distribution can be used to construct the probability distribution of the number of persons who will respond. For this problem, $n = 2$, $p = .4$ and $(1-p) = .6$. Substituting these values into the formula gives:

| X | | $P(X)$ |
|---|---|---|
| 0 | $P(0) = \binom{2}{0}(.4)^0(.6)^{2-0} = (.6)^2 =$ | .36 |
| 1 | $P(1) = \binom{2}{1}(.4)^1(.6)^{2-1} = 2(.4)(.6) =$ | .48 |
| 2 | $P(2) = \binom{2}{2}(.4)^2(.6)^{2-2} = (.4)^2 =$ | .16 |
| | | 1.00 |

Reading from the probability distribution, we see, for instance, that the probability that no person will respond is .36.

The mean and standard deviation of the above probability distribution are $\mu = (2)(.4) = .8$ and $\sigma = \sqrt{(2)(.4)(.6)} = .69$.

A graph of the binomial probability distribution with $p = .4$ and $n = 2$ is shown in Figure 7.1.

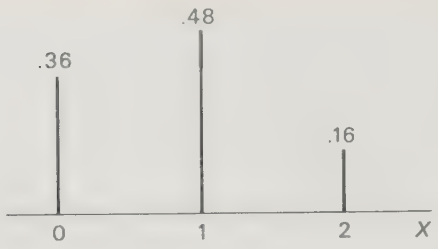



Figure 7.1

**Problem 1**

The probability that a part is defective is .2, and whether one part is defective or not is independent of the status of any other part.

(a) Using the formula for the binomial distribution, obtain the probability distribution of the number of defective parts in a group of three. Plot the distribution you obtain. √

(b) Is the distribution symmetrical or skewed? _____

(c) Obtain the mean and standard deviation of the probability distribution developed in (a).
$\mu =$ _____ ; $\sigma =$ _____

(d) What is the probability that at least one of the three parts will be defective? _____
_____ √

**Problem 2**

An auditor is reviewing the vouchers of a firm. The probability that a voucher contains an error is .001, and the error status of a voucher is independent of the status of any other voucher.

(a) Use the formula for the binomial probability distribution to obtain the probability that no erroneous vouchers will be found in a review of 1000 vouchers selected at random. (*Hint*: Use logarithms.)
_____

(b) What is the expected number of erroneous vouchers in the review of 1000 vouchers?
_____ √

Example 2 In some problems dealing with the binomial probability distribution, we are interested in the proportion $X/n$ rather than $X$ itself. For instance, in Example 1, we may be interested in the *proportion* of people who respond to the questionnaire rather than in the actual number who respond. The probabilities for the different values $X/n$ are the same as for the corresponding values of $X$. To illustrate this, the binomial probability distribution in Example 1 is restated on page 61 in terms of $X/n$ rather than $X$.

| $X/n$ | $P(X/n) = P(X)$ |
|---|---|
| 0 | .36 |
| 1/2 | .48 |
| 1 | .16 |
| | 1.00 |

Thus, the probability that one half the people will respond is .48, that is, $P(X/n = 1/2) = P(X = 1) = .48$.

In subsequent discussion, the proportion $X/n$ will be denoted by the symbol $\bar{p}$ (read: $p$ bar). The mean of the probability distribution of $\bar{p}$ is $p$, and the standard deviation is $\sqrt{p(1-p)/n}$.

**Problem 3**

(a) Given $E(X) = np$, show that $E(\bar{p}) = p$. _____

(b) Given that $\sigma_X^2 = np(1-p)$, show that $\sigma_{\bar{p}}^2 = p(1-p)/n$. _____

_____ √

**Problem 4**

Consider the binomial probability distribution constructed in Problem 1(a). Restate this probability distribution in terms of the proportion of the group of three parts that are defective. √

Example 3 Tables containing binomial probabilities are available. Table A-4 is one such table. It provides binomial probabilities for $\bar{p}$, for selected values of $p$ and $n$. For instance, if $p = .30$ and $n = 5$, the probability that $\bar{p} = 4/5$ is .0284. This probability is read directly from Table A-4.

**Problem 5**

(a) Using Table A-4, obtain the probability distribution of $\bar{p}$ for $p = .45$ and $n = 3$. √

(b) If $\bar{p}$ is the proportion defective in the sample, what is $1 - \bar{p}$? _____

(c) If $p$ is the probability of a defective, what is $1 - p$? _____

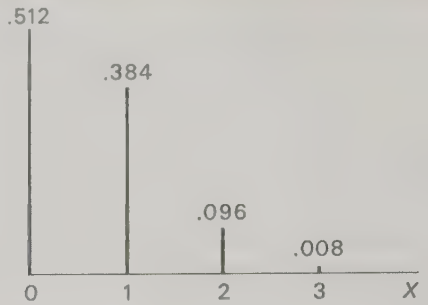
(d) If, when $n = 3$, $P(\bar{p} = 2/3 \mid p = .4) = .2880$, what is $P(1 - \bar{p} = 1/3 \mid 1 - p = .6)$? _____ √

(e) Using the principle demonstrated in (d), find from Table A-4 the probability that $\bar{p} = 6/7$ when $p = .6$ and $n = 7$. _____ √

**Answers**

**1. (a)**

| $X$ | | $P(X)$ |
|---|---|---|
| 0 | $P(0) = \binom{3}{0}(.2)^0(.8)^3 =$ | .512 |
| 1 | | .384 |
| 2 | | .096 |
| 3 | | .008 |
| | | 1.000 |




(b) The binomial distribution is skewed for all $p$ values except $p = .5$. The distribution in (a) has $p = .2$ and is skewed to the right.

(c) $\mu = np = 3(.2) = .60$; $\sigma = \sqrt{np\,(1-p)} = \sqrt{3(.2)(.8)} = .69$.

(d) $P(X = 1) + P(X = 2) + P(X = 3) = 1 - P(X = 0) = 1 - .512 = .488$.

**2. (a)** $P(X = 0) = \binom{1000}{0}(.001)^0(.999)^{1000-0} = (.999)^{1000} = .37$.
Note: $\log_{10} P(X = 0) = 1000 \log_{10} .999 = -.43$ and $\text{antilog}_{10}(-.43) = .37$.

(b) $\mu = np = 1000\,(.001) = 1$.

**3. (a)** $E(\bar{p}) = E(X/n) = (1/n)E(X) = (1/n)np = p$.

**(b)** Since $\bar{p} = X/n$, we have $\sigma^2_{\bar{p}} = (1/n)^2\,\sigma^2_X = (1/n)^2\,np\,(1-p) = p(1-p)/n$.

**4.**

| $\bar{p} = X/n$ | $P(\bar{p}) = P(X)$ |
|---|---|
| 0 | .512 |
| 1/3 | .384 |
| 2/3 | .096 |
| 1 | .008 |
| | 1.000 |

**5. (a)**

| $\bar{p}$ | $P(\bar{p})$ |
|---|---|
| 0 | .1664 |
| 1/3 | .4084 |
| 2/3 | .3341 |
| 1 | .0911 |
| | 1.0000 |

**(b) Proportion not defective in the sample.**

**(c) Probability of a nondefective.**

**(d)** .2880.

**(e)** $P(\bar{p} = 6/7 \mid p = .6) = P(1 - \bar{p} = 1/7 \mid 1 - p = .4) = .1306$.
Note that Table A-4 considers only $p$ values up to .5. For $p$ values greater than .5, we obtain the correct probabilities by looking up $1 - \bar{p}$ using the binomial parameter $1 - p$ and the same $n$ value.

**B. Normal Probability Distribution.** The random variable associated with a normal distribution is a continuous one, since it can assume any value on a scale. The two symmetrical, bell-shaped curves shown in Figure 7.2 are examples of normal distributions.

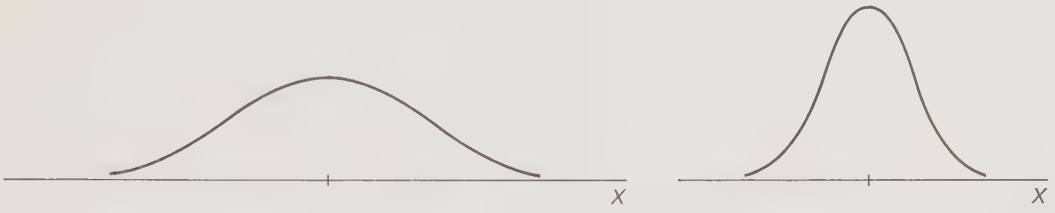



**Figure 7.2**

The formula for the curve of the normal probability distribution is:

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(X-\mu)^2}{\sigma^2}}$$

where $f(X)$ is the ordinate of the curve at $X$, $\pi$ is a constant (about 3.142), $e$ is a constant (about 2.718), and $\mu$ and $\sigma$ are respectively the mean and standard deviation of the random variable.

Areas under a continuous probability distribution correspond to probabilities. Table A-1 contains areas for the *standard normal distribution*, that is, for the normal distribution with a mean of zero and a standard deviation of one. This table can be used for any normal distribution, since every such distribution can be reduced to the standard normal distribution by applying the transformation $z = (X - \mu)/\sigma$. The variable $z$ has a standard normal distribution and frequently is called a *standard normal deviate.*

Example 4 The revenue a firm will realize in the coming year is considered to be a normal random variable, with a mean of \$525 million and a standard deviation of \$20 million. Let $X$ denote the revenue.

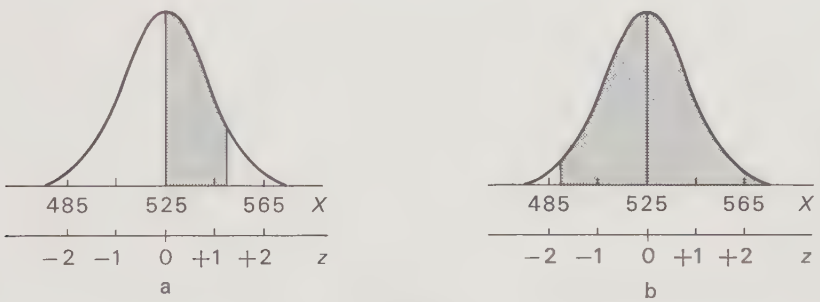



**Figure 7.3**

(a) It is desired to determine the probability that revenue will be between \$525 million and \$550 million. To obtain this probability, we must find the area shaded in Figure 7.3a. The $X$ values 525 and 550 correspond to $z$ values of $(525 - 525)/20 = 0$ and $(550 - 525)/20 = 1.25$ respectively. Recall that $z = (X - \mu)/\sigma$. Table A-1 gives the area under the normal curve *between the mean ($z = 0$) and any other specified z value.* Looking in the table for $z = 1.25$, we find .3944. Hence the probability that revenue will be between \$525 million and \$550 million is .3944.

(b) It is desired to determine the probability that revenue will exceed \$490 million. Hence, we must determine the area shaded in Figure 7.3b. This may be done in two steps: first, by determining the probability that revenue will fall between 490 and the mean (525); second, by determining the probability that revenue will exceed the mean. For the first probability, the $X$ value of 490 corresponds to the $z$ value $(490 - 525)/20 = -1.75$. Since areas to the left of the mean correspond to those to the right, we look up $z = 1.75$ in Table A-1 to obtain .4599. The second probability is .5 because half of the normal distribution is to the right of the mean (525). Combining the two probabilities, we find the probability that revenue will exceed \$490 million is .4599 + .5000 = .9599.

**Problem 6**

Refer to Example 4. Obtain the probability that revenue will be:

(a) between \$500 million and \$565 million _____

(b) greater than \$560 million _____

(c) less than \$545 million _____

(d) less than \$565 million _____

(e) From (c) and (d), what is $P(545 \leq X \leq 565)$? _____ √

**Problem 7**

In a can-filling operation, the weight of fill in the cans is normally distributed with a mean of 21.3 ounces and a standard deviation of .5 ounces.

(a) The can label states that the weight of the fill is 20 ounces. What per cent of the filled cans will be underweight? _____

(b) Find the 80th percentile weight. _____ √

Example 5 The sum (or difference) of two statistically independent normal random variables is also normally distributed. This can be generalized to more than two independent normal variables.

**Problem 8**

The lengths of time (in minutes) required to fill and pack a customer's order are statistically independent normal random variables, with means and standard deviations:

| Operation | Mean | Standard Deviation |
|---|---|---|
| Fill | 20 | 4 |
| Pack | 12 | 3 |

(a) What are the mean and standard deviation of the total time required to pack and fill a customer's order? _____ _____

(b) What is the form of the probability distribution of total time? _____

(c) What is the probability that the total time required to pack and fill a customer's order will exceed 40 minutes? _____ √

**C. Poisson Probability Distribution.** The Poisson probability distribution is described by the following formula:

$$P(X) = \frac{(\mu)^X e^{-\mu}}{X!} \quad \text{for } X = 0, 1, 2, \cdots \text{ ad infinitum.}$$

Here $e$ is a constant (about 2.718). The mean and standard deviation of the Poisson distribution are $\mu$ and $\sigma = \sqrt{\mu}$ respectively.

Example 6 Suppose that the number of paint blisters per square foot of a painted surface, denoted by $X$, follows a Poisson distribution with $\mu = .2$. We obtain the first five terms of the probability distribution for the number of paint blisters per square foot as follows (note that $e^{-.2} = .8187$):

| $X$ | $P(X)$ |
|---|---|
| 0 | $\frac{(.2)^0(.8187)}{0!} = .8187$ |
| 1 | $\frac{(.2)^1(.8187)}{1!} = .1637$ |
| 2 | $\frac{(.2)^2(.8187)}{2!} = .0164$ |
| 3 | $\frac{(.2)^3(.8187)}{3!} = .0011$ |
| 4 | $\frac{(.2)^4(.8187)}{4!} = .0001$ |

Thus, under the conditions assumed, there is a probability of .8187 that a square foot of the painted surface will contain no paint blisters, and so on.

The mean and standard deviation of the above probability distribution are $\mu = .2$ and $\sigma = \sqrt{.2} = .45$.

A graph of the Poisson distribution with $\mu = .2$ is shown in Figure 7.4.

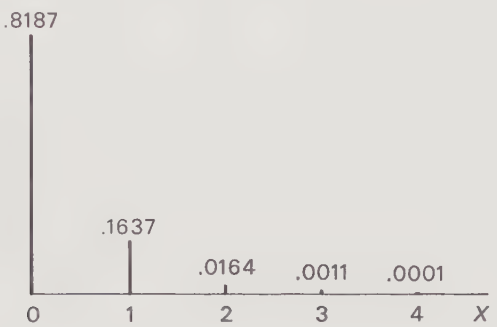



**Figure 7.4**

**Problem 9**

In Figure 7.4, why is $P(X = 5)$ not shown? _____ √

**Problem 10**

The number of customers who arrive each minute at a supermarket follows a Poisson distribution with $\mu = .5$.

(a) What is the probability that two customers will arrive in a given minute? (Note: $e^{-.5} = .6065$)

_____

(b) What is the standard deviation of this probability distribution? _____ √

Example 7 Tables such as Table A-2 are available providing Poisson probabilities for a large number of values of $\mu$. For instance, to obtain the solution to Problem 10, we consult Table A-2 for $\mu = .5$ to obtain $P(X = 2) = .0758$.

**Problem 11**

(a) Using Table A-2, construct the Poisson probability distribution for $\mu = .3$. Plot the distribution. √

(For answer, see page 68.)

(b) Is the distribution symmetrical or skewed? _____ √ (For answer, see page 68.)

**Answers**

6. (a) $X = 500$; $z = (500 - 525)/20 = -1.25$;
$X = 565$; $z = +2.00$.
Probability = .3944 + .4772 = .8716.

(b) $X = 560$; $z = +1.75$.
Probability = .5000 − .4599 = .0401.

(c) $X = 545$; $z = +1.00$.
Probability = .5000 + .3413 = .8413.

(d) $X = 565$; $z = +2.00$.
Probability = .5000 + .4772 = .9772.

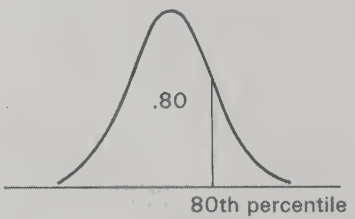
(e) $P(545 \leqslant X \leqslant 565) = P(X \leqslant 565) - P(X \leqslant 545) = .9772 - .8413 = .1359$.

7. (a) $X = 20$; $z = (20 - 21.3)/.5 = -2.60$.
Probability = .5000 − .4953 = .0047.
Hence, .47% of the filled cans will be underweight.

(b) The $z$ value corresponding to the 80th percentile is +.84.
From $z = (X - \mu)/\sigma$, it follows that:

$$+.84 = \frac{(\text{80th percentile weight}) - 21.3}{.5}$$

80th percentile weight = 21.3 + .84(.5) = 21.72 ounces.




8. (a) **Mean of the total time = 20 + 12 = 32 minutes.**
**Variance of the total time = $(4)^2 + (3)^2 = 25$.**
**Standard deviation of the total time = $\sqrt{25} = 5$ minutes.**

(b) **Normal.**

(c) $X = 40$; $z = (40 - 32)/5 = +1.60$.
Probability = .5000 − .4452 = .0548.

9. $P(X = 5)$ is not shown because it is a very small value (less than .0001).

10. (a) $P(X = 2) = \dfrac{(.5)^2\, e^{-.5}}{2!} = \dfrac{(.5)^2\,(.6065)}{2!} = .0758$.

(b) $\sigma = \sqrt{.5} = .707$.

**Problem 12**

The number of automobiles waiting for the next trip of an automobile ferry follows a Poisson distribution with $\mu = 7$. Determine from Table A-2:

(a) The automobile capacity the ferry should have in order that the probability is less than .02 that all the automobiles will not be accommodated. _____

(b) The probability that the ferry will travel empty. _____ √

**D. Exponential Probability Distribution.** The exponential probability distribution is a continuous probability distribution with the following shape:

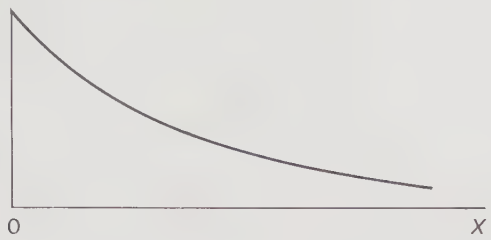



**Figure 7.5**

The formula for the exponential probability curve is:

$$f(X) = \frac{1}{\mu} e^{-X/\mu}$$

where $e$ is a constant (about 2.718). The mean and standard deviation of the exponential distribution are $\mu$ and $\sigma = \mu$ respectively.

Example 8 Tables are available providing the exponential probabilities for the *right tail of the distribution*. Table A-3 is one such table.

Assume that the service life of an electronic component is described by an exponential distribution with $\mu = 700$ hours. The component is to be used on a job which will last 140 hours. To determine the probability that the component will last at least until the job is completed, we need the multiple $X/\mu$. Here it is $X/\mu = 140/700 = .2$. Looking up this multiple in Table A-3, we find the probability is .819 that the component will last longer than 140 hours.

**Problem 13**

For the electronic component in Example 8, determine the probability that the service life:

(a) will exceed 1330 hours _____

(b) will be less than 700 hours _____

(c) will be between 630 and 1120 hours. _____

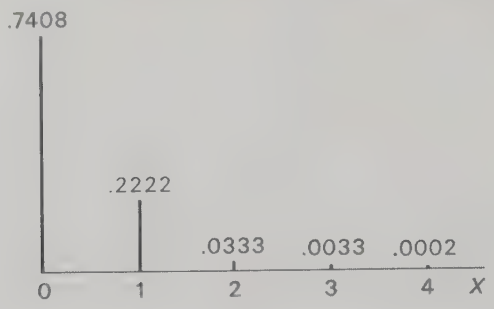
(d) Find the median service life of the component. _____

(e) What is the standard deviation of the service life? _____ √

**Answers**

**11.** (a)

| $X$ | $P(X)$ |
|---|---|
| 0 | .7408 |
| 1 | .2222 |
| 2 | .0333 |
| 3 | .0033 |
| 4 | .0002 |
| | 1.0000 |




**(b) The Poisson distribution is always skewed. Hence, the distribution in (a) is skewed.**

**12.** (a) For $\mu = 7$, $P(X > 13) = P(X = 14) + P(X = 15) + \cdots = .0071 + .0033 + \cdots = .0127$, and similarly, $P(X > 12) = .0269$. Thus a capacity of 13 automobiles meets the specification.

(b) $P(X = 0) = .0009$.

**13.** (a) $X/\mu = 1330/700 = 1.9$; $P(X > 1330) = .150$.

(b) $X/\mu = 700/700 = 1.0$; $P(X < 700) = 1 - P(X \geqslant 700) = 1 - .368 = .632$.

(c) $X/\mu = 630/700 = .9$ and $1120/700 = 1.60$, $P(X > 630) = .407$ and $P(X > 1120) = .202$. Hence $P(630 \leqslant X \leqslant 1120) = .407 - .202 = .205$.

(d) $P(X > \text{median}) = .5$. Interpolating in Table A-3, we have $X/\mu = \text{median}/\mu = .69$ corresponds to a probability of .5. Thus, median $= .69\mu = .69(700) = 483$ hours.

(e) $\sigma = \mu = 700$ hours.

## REVIEW PROBLEMS

1. In a marketing experiment, a man is shown a pair of belts, a pair of shirts, and a pair of suits, and in each case is asked to identify the more expensive item of the two. Use the binomial distribution and assume that the person has a probability of .5 of identifying the more expensive item correctly (i.e., he is guessing).
   (a) What is the probability that the person will make correct identifications in all three pairs? _____
   (b) What is the mean number of correct identifications that will be made? _____
   (c) Construct and plot the probability distribution of the number of correct identifications. √

2. A mail order crew can fill 1000 orders in an eight-hour day, and will work overtime to fill all orders the day they are received. All orders for the day are received first thing in the morning. If the number of orders received each day is normally distributed with a mean of 750 orders and a standard deviation of 250 orders, then:
   (a) What is the probability that overtime work will be required to complete the orders on any given day?
   (b) What is the probability according to the normal probability distribution that the number of orders to be filled in a given day is less than zero? _____ Is this result operationally meaningful? _____
   What does this result suggest about the validity of using the normal distribution in this problem? √

3. A cable manufacturer obtains large shipments of 500-foot lengths of wire from a supplier. Any shipment where less than 4% of the wire lengths have three or more defects per length is acceptable. If the number of defects per length of wire in a shipment follows a Poisson distribution with a mean of .5 defects per length, does that shipment meet the acceptance specifications of the manufacturer? _____ Why or why not? √

4. The duration of telephone calls follows an exponential distribution with $\mu = 2$ minutes.
   (a) What proportion of calls will be less than three minutes long? _____
   (b) 95% of the calls last less than how many minutes? _____ √

## ANSWERS TO REVIEW PROBLEMS

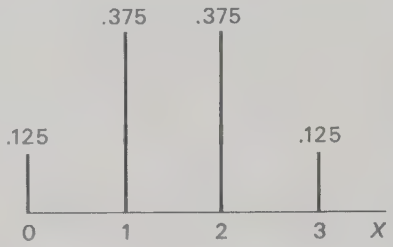
1. Binominal distribution with $p = .5$ and $n = 3$:

   (a) $P(X = 3) = \binom{3}{3}(.5)^3(.5)^{3-3} = .125$.

   (b) $\mu = np = 3(.5) = 1.5$.

   (c)

| $X$ | $P(X)$ |
|---|---|
| 0 | .125 |
| 1 | .375 |
| 2 | .375 |
| 3 | .125 |
| | 1.000 |




2. (a) $P(X > 1000) = P(z > 1) = .5000 - .3413 = .1587$, where $z = (1000 - 750)/250 = 1$.

   (b) $P(X < 0) = P(z < -3) = .5000 - .49865 = .00135$, where $(0 - 750)/250) = -3$.

   **"Negative" numbers of orders generally will not be operationally meaningful. The fact that a negative number of orders is possible indicates that the normal probability distribution is an approximation to the actual probability distribution. Whether or not the approximation is reasonable depends on the circumstances of the problem.**

3. When $\mu = .5$, $P(X \geq 3) = .0126 + .0016 + .0002 + \cdots = .0144$. Hence with $\mu = .5$, 1.44% of the wire lengths will have three or more defects. This meets the specification of "less than 4%."

4. (a) $X/\mu = 3/2 = 1.5$; $P(X < 3) = 1 - P(X \geq 3) = 1 - .223 = .777$.
   (b) $P(X <$ 95th percentile$) = .95$. In Table A-3, we have $X/\mu =$ 95th percentile$/\mu = 3.0$ corresponds to a right tail area of .05. Thus, 95th percentile $= 3.0\mu = 3.0(2) = 6.0$ minutes.

# 8

# Introduction to Sampling Distributions

Frequently one must rely on sampling to provide information about the population of interest. The concept of a sampling distribution plays a central role in drawing inferences from a sample about the population.

**A. Definitions.** A *population* is the totality of items under study. A population may contain a *finite* or *infinite* number of items. Infinite populations are associated with processes. Consider the amount of coffee issued per cup by an automatic vending machine. The infinite population associated with this process refers to the pattern of variation in the amount of coffee issued per cup if the vending machine continues to pour cups of coffee indefinitely under existing conditions.

A *sample* is a part of the population under study selected so that inferences can be made from it about the entire population. Characteristics of a population are called *parameters* (e.g., population mean), while characteristics of a sample are called *statistics* (e.g., sample mean).

**B. Sampling.**

Example 1 To illustrate sampling, consider a finite population composed of five individuals, $A, B, C, D$, and $E$. Suppose a sample of three individuals is to be selected from this population. We shall enumerate all combinations of three individuals that could be selected for the sample. Note that this permits any individual to be selected only once for a sample. This is called *sampling without replacement*. The possible sample combinations are:

| | |
|---|---|
| $A, B, C$ | $A, D, E$ |
| $A, B, D$ | _____ |
| $A, B, E$ | _____ |
| $A, C, D$ | _____ |
| $A, C, E$ | _____ |

*Sampling with replacement* is occasionally used. It differs in that the selected individual is returned to the population before the next individual is selected for the sample, and thus an individual may appear more than once in the sample.

**Problem 1**

Fill in the blanks for the remaining four sample combinations in Example 1. √

**C. Simple Random Sampling.** In order to use the analytical tools of statistical inference, it is necessary that the sample elements be selected according to known probabilities. The most basic type of sample in this category is a *simple random sample*, often simply called a *random sample*.

Simple random sampling takes two forms. First, if sampling is from a finite population without replacement, then a simple random sample requires that *each possible sample combination have an equal probability of being chosen as the sample*. Second, if sampling is from an infinite population (or from a finite population when sampling is with replacement), then a simple random sample requires that the *sample observations* that make up the sample *be statistically independent*.

**Answers**

1. *B, C, D,*
   *B, C, E,*
   *B, D, E,*
   *C, D, E.*

**Problem 2**

Consider the sampling problem of Example 1. If simple random sampling without replacement is used, what is the probability of choosing any one of the ten sample combinations? _____ √

**D. Parameter and Statistic of Interest.**

Example 2 Consider the population of five individuals in Example 1. Assume that the incomes of these individuals (in $ thousands) are: $X = 5, 5, 7, 10,$ and 8 respectively. Suppose the parameter of interest in this population is the total income of the population, denoted by $\tau$. Here $\tau = 5 + 5 + 7 + 10 + 8 = 35$. Further, suppose that the value of $\tau$ is unknown and is to be estimated from a simple random sample of three individuals. The statistic to be used to estimate $\tau$ is the total income of the three individuals in the sample, multiplied by the ratio 5/3. Denote this statistic by $T$. Hence:

$$T = 5/3\ (X_1 + X_2 + X_3).$$

Here $X_1$, $X_2$, and $X_3$ are the incomes of the individuals in the sample. For instance, if the sample contained individuals $A$, $B$, and $E$ with incomes of 5, 5, and 8 respectively, then the value of the statistic $T$ for this particular sample is $T = 5/3(5 + 5 + 8) = 30$. The ratio 5/3 is used because it is the ratio of the population size to the sample size.

**Problem 3**

If the sample combination consists of individuals $A$, $C$, and $D$, what is the value of $T$? _____
_____ √

**Problem 4**

Consider the population of five individuals and their incomes, 5, 5, 7, 10, and 8, given in Example 2. Suppose the parameter of interest is the median income of the population.

(a) The population median income in this problem is? _____ √

(b) Suppose the incomes of a simple random sample of three individuals are to be used to estimate the population median income. What sample statistic would you suggest using as an estimator of the population median income? _____ √

(c) If the sample contains individuals $B$, $D$, and $E$, the value of the sample statistic is? _____ √

**E. Sampling Distribution.** A sampling distribution is a probability distribution where the possible outcomes are the different values a sample statistic may assume, given that simple random sampling is used.

Below we will demonstrate how a sampling distribution is obtained when the population is finite and simple random sampling without replacement is used. The construction of sampling distributions for infinite populations (or for finite populations if sampling is with replacement) involves complications and will not be discussed here.

Example 3 Let us continue with the problem from Example 2 dealing with the estimation of the total income of the population. Ten possible samples of three individuals may be obtained. These were enumerated in Example 1. Simple random sampling ensures that each combination has a probability of .1 of being selected. The $T$ statistic that would be calculated for each of the ten possible samples is shown in Table 8.1.

**Table 8.1**

| Sample | | | |
|---|---|---|---|
| Individuals | Incomes | $T = 5/3\ (X_1 + X_2 + X_3)$ | Probability of Sample |
| *A, B, C* | 5, 5, 7 | 5/3 (5 + 5 + 7) = 28.33 | .1 |
| *A, B, D* | 5, 5, 10 | 33.33 | .1 |
| *A, B, E* | 5, 5, 8 | 30.00 | .1 |
| *A, C, D* | 5, 7, 10 | 36.67 | .1 |
| *A, C, E* | 5, 7, 8 | 33.33 | .1 |
| *A, D, E* | 5, 10, 8 | 38.33 | .1 |
| *B, C, D* | 5, 7, 10 | 36.67 | .1 |
| *B, C, E* | _____ | _____ | ___ |
| *B, D, E* | _____ | _____ | ___ |
| *C, D, E* | _____ | _____ | ___ |

**Problem 5**

Complete Table 8.1. √

Example 4 The sampling distribution of the statistic $T$ is obtained by combining the values of $T$ in Table 8.1 into a probability distribution. For example, a value of $T = 38.33$ arises twice and thus has a probability of .1 + .1 = .2 of occurring. The sampling distribution of $T$ is:

| $T$ | $P(T)$ |
|---|---|
| 28.33 | .1 |
| 30.00 | .1 |
| 33.33 | .3 |
| 36.67 | .2 |
| 38.33 | .2 |
| 41.67 | .1 |
| | 1.0 |

**Problem 6**

For the population of five individuals and their incomes, 5, 5, 7, 10, and 8, we wish to obtain the sampling distribution of $T$ for a simple random sample of size two.

(a) Indicate the form of the $T$ statistic when the sample size is two. _____ √

(For answer, see page 76.)

(b) Obtain the sampling distribution of $T$ for a simple random sample (without replacement) of size two. √

(For answer, see page 76.)

| Sample | | | | Sampling Distribution | |
|---|---|---|---|---|---|
| Individuals | Incomes | $T$ | Probability | $T$ | $P(T)$ |

**Answers**

2. Since each of the ten sample combinations must be equally likely, each has a probability of 1/10 = .1 of being chosen.

3. $T = 5/3(5 + 7 + 10) = 36.67$.

4. **(a) Population median income = $7 thousand.**
   **(b) The sample median seems a logical candidate for estimating the population median income.**
   (c) The incomes of *B*, *D*, and *E* are 5, 10, and 8 respectively. The sample median therefore is 8.

5.

| Incomes | | *T* | Probability |
|---|---|---|---|
| 5, 7, 8 | 5/3(5 + 7 + 8) = | 33.33 | .1 |
| 5, 10, 8 | | 38.33 | .1 |
| 7, 10, 8 | | 41.67 | .1 |

Example 5 The sampling distributions for the statistic *T* obtained in Example 4 and Problem 6 are shown in Table 8.2.

**Table 8.2**

| Sample of Size Two | | Sample of Size Three | |
|---|---|---|---|
| *T* | *P(T)* | *T* | *P(T)* |
| 25.0 | .1 | 28.33 | .1 |
| 30.0 | .2 | 30.00 | .1 |
| 32.5 | .2 | 33.33 | .3 |
| 37.5 | .3 | 36.67 | .2 |
| 42.5 | .1 | 38.33 | .2 |
| 45.0 | .1 | 41.67 | .1 |
| | 1.0 | | 1.0 |

The total income of the population is $\tau = 35$. The sampling distributions in Table 8.2 provide information on the precision of *T* as an estimator of $\tau$. For instance, with a sample of size three, no sample will lead to a value of *T* which differs by more than 6.67 from $\tau = 35$, that is, the possible error due to sampling is at most about (6.67/35) × 100 = 19.1%.

**Problem 7**

Consider the two sampling distributions in Example 5.

(a) For each of the sampling distributions, determine the probability that *T* will differ from $\tau = 35$ by ±$5 thousand or less. _____ _____

(b) Compare the two probabilities in (a). What is the effect of sample size on estimation precision?

_____ √

**Problem 8**

(a) Calculate the mean of the sampling distribution for a sample of size two in Example 5.

_____

(b) The mean of the sampling distribution for a sample of size three is 35. Compare these mean values with the value of $\tau$. _____ √

Example 6 Note that the mean of the sampling distribution of $T$ is equal to the value of $\tau$ for both sample sizes. Where the sampling distribution of a statistic ($T$ in our case) has a mean equal to the parameter being estimated ($\tau$ in our case), the statistic is said to be an *unbiased estimator* of the parameter.

**Problem 9**

Calculate the standard deviation of the sampling distribution for a sample of size two in Example 5. _____ _____ The standard deviation of the sampling distribution for a sample of size three is 3.87. What implication may be drawn from a comparison of these two standard deviations? _____ _____ √

Example 7 Consider the population of five individuals and their incomes, 5, 5, 7, 10, and 8. In Problem 4, the parameter of interest was the population median income, and the sample median was to be used to estimate the population median income. Derived in Table 8.3 is the sampling distribution of the sample median for a simple random sample of size three.

**Table 8.3**

| Sample | | | |
|---|---|---|---|
| Individuals | Incomes | Sample Median Income | Probability of Sample |
| *A, B, C* | 5, 5, 7 | 5 | .1 |
| *A, B, D* | 5, 5, 10 | 5 | .1 |
| *A, B, E* | 5, 5, 8 | 5 | .1 |
| *A, C, D* | 5, 7, 10 | 7 | .1 |
| *A, C, E* | 5, 7, 8 | 7 | .1 |
| *A, D, E* | 5, 10, 8 | 8 | .1 |
| *B, C, D* | 5, 7, 10 | 7 | .1 |
| *B, C, E* | 5, 7, 8 | 7 | .1 |
| *B, D, E* | 5, 10, 8 | 8 | .1 |
| *C, D, E* | 7, 10, 8 | 8 | .1 |

| Sampling Distribution of Sample Median | |
|---|---|
| Sample Median | *P*(Sample Median) |
| 5 | .3 |
| 7 | .4 |
| 8 | .3 |
| | 1.0 |

The mean and standard deviation of the sampling distribution in Table 8.3 are 6.7 and 1.19 respectively.

**Problem 10**

(a) Is the sample median in Table 8.3 an unbiased estimator? _____

(b) What is the probability that the sample median differs by more than ±$1 thousand from the population median? _____ √

Answers

**6.** (a) $T = 5/2\,(X_1 + X_2)$ where $X_1$ and $X_2$ are the incomes of the two individuals in the sample.

(b)

| Sample | | | | Sampling Distribution | |
|---|---|---|---|---|---|
| Individuals | Incomes | $T$ | Probability | $T$ | $P(T)$ |
| $AB$ | 5, 5 | 25.0 | .1 | 25.0 | .1 |
| $AC$ | 5, 7 | 30.0 | .1 | 30.0 | .2 |
| $AD$ | 5, 10 | 37.5 | .1 | 32.5 | .2 |
| $AE$ | 5, 8 | 32.5 | .1 | 37.5 | .3 |
| $BC$ | 5, 7 | 30.0 | .1 | 42.5 | .1 |
| $BD$ | 5, 10 | 37.5 | .1 | 45.0 | .1 |
| $BE$ | 5, 8 | 32.5 | .1 | | 1.0 |
| $CD$ | 7, 10 | 42.5 | .1 | | |
| $CE$ | 7, 8 | 37.5 | .1 | | |
| $DE$ | 10, 8 | 45.0 | .1 | | |

**7.** (a)

| Sample Size | $P(30 \leqslant T \leqslant 40)$ |
|---|---|
| 2 | .7 |
| 3 | .8 |

Note: $\tau + 5 = 40$; $\tau - 5 = 30$.

(b) In (a), the probability that $T$ will be within \$5 thousand of $\tau$ increases from .7 to .8 as the sample size increases from 2 to 3. This demonstrates that estimation precision increases with sample size.

**8.** (a) $\mu_T = 25.0(.1) + 30.0(.2) + 32.5(.2) + 37.5(.3) + 42.5(.1) + 45.0(.1) = 35.$

(b) Both sampling distributions have means equal to $\tau$, that is, $\mu_T = \tau = 35$.

**9.** $\sigma_T^2 = (25.0 - 35)^2(.1) + (30.0 - 35)^2(.2) + (32.5 - 35)^2(.2) + (37.5 - 35)^2(.3) + (42.5 - 35)^2(.1) + (45.0 - 35)^2(.1) = 33.75;$

$\sigma_T = \sqrt{33.75} = 5.81.$

**The standard deviation of the sampling distribution represents a measure of precision of the estimate obtained from the sample. We see below that the standard deviation becomes smaller as the sample size increases.**

| Sample Size | $\sigma_T$ |
|---|---|
| 2 | 5.81 |
| 3 | 3.87 |

**10.** (a) **No. Since the mean of the sampling distribution does not equal the population median income of 7, the sample median is not an unbiased estimator here.**

(b) $P$(sample median $< 6$ or sample median $> 8$) = .3.
*Note:* population median + 1 = 7 + 1 = 8;
population median −1 = 7 −1 = 6.

## REVIEW PROBLEMS

1. (a) Refer to Example 7. Obtain the sampling distribution of the sample median for a simple random sample of size four. (Remember that the median of an even number of items is equal to the average of the two middle values.) √

   (b) Calculate the mean and standard deviation of the sampling distribution obtained in (a). _____ _____ _____ √

   (c) Is the sample median an unbiased estimator of the population median income? _____ Explain. _____ √

2. Compare the standard deviations of the sampling distributions of the sample median derived in Example 7 and Review Problem 1. What is the relationship between precision and sample size? _____ _____ √

3. For the population in Example 7, construct the sampling distribution of the sample mean for samples of size one. √

## ANSWERS TO REVIEW PROBLEMS

1. (a)

| Sample | | | |
|---|---|---|---|
| Individuals | Incomes | Sample Median Income | Probability of Sample |
| *A, B, C, D* | 5, 5, 7, 10 | (5 + 7)/2 = 6.0 | .2 |
| *A, B, C, E* | 5, 5, 7, 8 | 6.0 | .2 |
| *A, B, D, E* | 5, 5, 10, 8 | 6.5 | .2 |
| *A, C, D, E* | 5, 7, 10, 8 | 7.5 | .2 |
| *B, C, D, E* | 5, 7, 10, 8 | 7.5 | .2 |

| Sampling Distribution of Sample Median | |
|---|---|
| Sample Median | *P*(Sample Median) |
| 6.0 | .4 |
| 6.5 | .2 |
| 7.5 | .4 |
| | 1.0 |

(b) Mean = $6.0(.4) + 6.5(.2) + 7.5(.4) = 6.7$;
variance = $(6.0 - 6.7)^2(.4) + (6.5 - 6.7)^2(.2) + (7.5 - 6.7)^2(.4) = .46$;
standard deviation = $\sqrt{.46} = .68$.

(c) No, because the mean of the sampling distribution of the sample median is not equal to the population median, that is, $6.7 \neq 7$.

2.

| Sample Size | Standard Deviation of Sampling Distribution |
|---|---|
| 3 | 1.19 (from Example 7) |
| 4 | .68 (from Review Problem 1) |

We see that precision increases (standard deviation decreases) as the sample size increases.

3.

| Sample Mean | *P*(Sample Mean) |
|---|---|
| 5 | .4 |
| 7 | .2 |
| 8 | .2 |
| 10 | .2 |
| | 1.0 |

This sampling distribution has outcomes that correspond to the values of the population.

# 9

# Sampling Distribution of $\overline{X}$

Any sample statistic based on a simple random sample is a random variable with an associated probability distribution called a sampling distribution. We now concentrate on one of these sampling distributions, namely the sampling distribution of $\overline{X}$, where $\overline{X}$ denotes the sample mean.

**A. Finite Population (Sampling Without Replacement).**

Example 1 An auditor desires to determine, by physical count for each product in inventory, the difference between the value of the recorded number of items and that of the actual number of items. This difference is denoted by $X$. For the sake of simplicity, assume that the inventory is composed of five products and the $X$ values are as follows:

| Product | $X$ ($ thousands) |
|---|---|
| *A* | 0 |
| *B* | −1 |
| *C* | 0 |
| *D* | 1 |
| *E* | 2 |

On the basis of a simple random sample of four products, the auditor desires to estimate $\mu$, the mean difference between the recorded and actual values for the population. The statistic that will be used to estimate $\mu$ is the sample mean, $\overline{X}$.

The sampling distribution of $\overline{X}$ for this problem is derived by enumerating all possible sample combinations. The initial steps are as follows:

| Sample | | $\overline{X}$ | Probability of Sample |
|---|---|---|---|
| *A, B, C, D* | 0, −1, 0, 1 | 0 | .2 |
| *A, B, C, E* | 0, −1, 0, 2 | .25 | .2 |
| *A, B, D, E* | 0, −1, 1, 2 | .50 | .2 |
| *A, C, D, E* | 0, 0, 1, 2 | .75 | .2 |
| *B, C, D, E* | −1, 0, 1, 2 | .50 | .2 |

**Problem 1**

(a) Obtain the sampling distribution of $\overline{X}$. $\surd$ (For answer, see page 82.)

(b) Find the mean and standard deviation of the five values in the population. $\mu$ = _____;
$\sigma$ = _____ √

(c) What is the probability that $\overline{X}$ differs from $\mu$ by more than ±$.25 thousand? _____ √

**B. Mean and Standard Deviation of Sampling Distribution of $\overline{X}$.** Let $\mu_{\overline{X}}$ and $\sigma_{\overline{X}}$ denote the mean and standard deviation respectively of the sampling distribution of $\overline{X}$. For a finite population and sampling without replacement, we have:

$$\mu_{\overline{X}} = \mu \text{ and } \sigma_{\overline{X}} = \sqrt{\frac{N-n}{N-1}} \cdot \frac{\sigma}{\sqrt{n}},$$

where $N$ is the population size, $n$ is the sample size, $\mu$ is the population mean, and $\sigma$ is the population standard deviation. The multiplier $\sqrt{(N-n)/(N-1)}$ is called the *finite correction factor.*

Example 2 Utilizing the relations given above, we know the mean of the sampling distribution is equal to the population mean. Thus, $\mu_{\overline{X}} = \mu = .40$. Second, substituting $N = 5$, $n = 4$, and $\sigma = 1.02$ into the formula for the standard deviation of the sampling distribution of $\overline{X}$ gives:

$$\sigma_{\overline{X}} = \sqrt{\frac{5-4}{5-1}} \cdot \frac{1.02}{\sqrt{4}} = .255.$$

**Problem 2**

(a) Verify by direct computation from the sampling distribution of $\overline{X}$ in Example 1 that the mean and standard deviation of this sampling distribution are $\mu_{\overline{X}} = .40$ and $\sigma_{\overline{X}} = .255$.

(b) Is $\overline{X}$ an unbiased estimator of $\mu$? _____ Why? _____ √

**Problem 3**

(a) For the population in Example 1, obtain the sampling distribution of $\overline{X}$ for a sample size of two. √

(b) Using the formulas for $\mu_{\overline{X}}$ and $\sigma_{\overline{X}}$, obtain the mean and standard deviation of the sampling distribution in (a). $\mu_{\overline{X}}$ = _____; $\sigma_{\overline{X}}$ = _____ √

(For answer, see page 83.)

**Problem 4**

Refer to Example 2 and Problem 3. How does the magnitude of $\sigma_{\overline{X}}$ change with an increase in the sample size $n$? _____ √ (For answer, see page 83.)

**C. Infinite Population (Or Finite Population — Sampling With Replacement).** For infinite populations (or for finite populations if sampling is with replacement), the mean and standard deviation of the sampling distribution of $\overline{X}$ are given by $\mu_{\overline{X}} = \mu$ and $\sigma_{\overline{X}} = \sigma/\sqrt{n}$. These results are the same as for a finite population when sampling without replacement, except that the finite correction factor is now omitted from $\sigma_{\overline{X}}$.

**Problem 5**

Calculate the value of the finite correction factor for a sample of size 10 from populations of the following sizes:

(a) $N = 101$ _____

(b) $N = 10{,}001$ _____

(c) What may we conclude about the value of the correction factor as $N$ approaches infinity? _____ √

(For answers, see page 83.)

**D. Central Limit Theorem.** The central limit theorem tells us the nature of the probability distribution that sampling distributions of $\overline{X}$ follow. The theorem states:

*For almost all populations, the sampling distribution of $\bar{X}$ is approximately normal if the simple random sample size is sufficiently large.*

When the population is finite and sampling is without replacement, the population size must be considerably larger than the sample size for the theorem to apply, but such is usually the case in practice.

What is a "sufficiently large" sample size? For some populations, $n > 4$ or 5 is sufficient, for others $n > 30$, and for some highly skewed populations $n$ needs to be large indeed.

A special situation exists when the population being sampled is normally distributed. In this case, the central limit theorem is not needed because the sampling distribution of $\bar{X}$ is exactly normal for all sample sizes.

Example 3 Consider the auditing problem described in Example 1. However, let the inventory consist of 5000 products. Suppose that the population mean and population standard deviation of the differences between recorded and actual values for the 5000 products are known to be $\mu = \$4$ thousand and $\sigma = \$12$ thousand respectively.

If a simple random sample of 400 products is inspected, the central limit theorem tells us that the sampling distribution of $\bar{X}$ will be approximately normal. From our knowledge about the sampling distribution of $\bar{X}$, we know that the mean and standard deviation of this distribution are (in $ thousands):

$$\mu_{\bar{X}} = \mu = 4 \text{ and } \sigma_{\bar{X}} = \sqrt{\frac{5000 - 400}{5000 - 1}} \cdot \frac{12}{\sqrt{400}} = .58.$$

An auditor is interested in knowing the probability that the $\bar{X}$-value obtained from the sample will not differ from the true mean value $\mu = 4$ by more than $1 thousand, that is, the auditor wishes to know $P(3 \leqslant \bar{X} \leqslant 5)$. This probability is obtained from the table of normal probabilities. Since $z = (3 - 4)/.58 = -1.72$ and $z = (5 - 4)/.58 = +1.72$, we have:

$$P(3 \leqslant \bar{X} \leqslant 5) = P(-1.72 \leqslant z \leqslant +1.72) = .4573 + .4573 = .9146.$$

**Problem 6**

Refer to Example 3.

(a) What is the probability that $\bar{X}$ will be within 10 per cent of the true value of $\mu$? _____

(b) Find $P(3 \leqslant \bar{X} \leqslant 5)$ if a simple random sample of 100 products is used. _____ √ (For answers, see page 83.)

(c) Compare the result in (b) with that of Example 3. What is the effect of decreasing $n$? _____

_____

(d) Find $P(3 \leqslant \bar{X} \leqslant 5)$ if $\sigma = 24$ rather than $\sigma = 12$. Assume a sample of size 400 is used. _____

(e) Compare this result with that in Example 3. What is the effect of a larger $\sigma$? _____

_____ √

(For answers, see page 84.)

**Problem 7**

The daily revenues of a toll bridge have a mean of $12,200, and a standard deviation of $1400.

(a) The toll revenues for a simple random sample of 49 days are obtained, and $\bar{X}$ is calculated. What are the mean and standard deviation of the sampling distribution of $\bar{X}$? (*Hint*: The population of interest is an infinite one.) _____ _____

(b) Assuming that the central limit theorem applies for the sample in (a), within what range will the sample mean fall 99% of the time? Use symmetrical limits around the population mean. _____

_____

(c) What must be the sample size if the standard deviation of the sampling distribution of $\bar{X}$ is to be $\sigma_{\bar{X}} = \$100$?

_____

(d) If it is assumed that the distribution of daily toll revenues is exactly normal, what is the nature of the sampling distribution of $\bar{X}$ for a sample of size four? _____

Give the mean and standard deviation of this sampling distribution. _____

_____ √ (For answers, see page 84.)

Answers

1. (a)

| $\bar{X}$ | $P(\bar{X})$ |
|---|---|
| 0 | .2 |
| .25 | .2 |
| .50 | .4 |
| .75 | .2 |
| | 1.0 |

(b) $\mu = \Sigma X/N = (0 - 1 + 0 + 1 + 2)/5 = .4$;

$\sigma^2 = \Sigma(X - \mu)^2/N = [(0 - .4)^2 + (-1 - .4)^2 + (0 - .4)^2 + (1 - .4)^2 + (2 - .4)^2]/5 = 1.04$;

$\sigma = \sqrt{1.04} = 1.02$.

(c) $P(\bar{X} < .15 \text{ or } \bar{X} > .65) = .4$. *Note:* $\mu + .25 = .4 + .25 = .65$;
$\mu - .25 = .4 - .25 = .15$.

2. (a)

| $\bar{X}$ | $P(\bar{X})$ | $\bar{X}P(\bar{X})$ | $(\bar{X} - \mu_{\bar{X}})^2 P(\bar{X})$ |
|---|---|---|---|
| 0 | .2 | $0(.2) = 0$ | $(0 - .40)^2(.2) = .0320$ |
| .25 | .2 | .05 | .0045 |
| .50 | .4 | .20 | .0040 |
| .75 | .2 | .15 | .0245 |
| Total | 1.0 | $\mu_{\bar{X}} = .40$ | $\sigma^2_{\bar{X}} = .0650$, thus $\sigma_{\bar{X}} = .255$ |

(b) Yes, because the mean of the sampling distribution of $\bar{X}$ is equal to the population mean, that is, $\mu_{\bar{X}} = \mu$.

3. (a)

| Sample | | | $\bar{X}$ | Probability of Sample |
|---|---|---|---|---|
| *AB* | 0, | −1 | −.5 | .1 |
| *AC* | 0, | 0 | 0 | .1 |
| *AD* | 0, | 1 | .5 | .1 |
| *AE* | 0, | 2 | 1.0 | .1 |
| *BC* | −1, | 0 | −.5 | .1 |
| *BD* | −1, | 1 | 0 | .1 |
| *BE* | −1, | 2 | .5 | .1 |
| *CD* | 0, | 1 | .5 | .1 |
| *CE* | 0, | 2 | 1.0 | .1 |
| *DE* | 1, | 2 | 1.5 | .1 |

Sampling Distribution of $\bar{X}$

| $\bar{X}$ | $P(\bar{X})$ |
|---|---|
| − .5 | .2 |
| 0 | .2 |
| .5 | .3 |
| 1.0 | .2 |
| 1.5 | .1 |
| | 1.0 |

**3.(cont'd.)**

(b) $\mu_{\bar{X}} = .40$; $\sigma_{\bar{X}} = \sqrt{\frac{5-2}{5-1}} \cdot \frac{1.02}{\sqrt{2}} = \sqrt{.39} = .62$.

**4.** $\sigma_{\bar{X}}$ decreases as $n$ increases. This fact is demonstrated by the results obtained in Example 2 and Problem 3:

| $n$ | $\sigma_{\bar{X}}$ |
|---|---|
| 2 | .62 (Problem 3) |
| 4 | .255 (Example 2) |

This can also be seen from the formula for $\sigma_{\bar{X}}$.

**5.** (a) $\sqrt{(101-10)/(101-1)} = .9539$.

(b) .9995.

(c) As $N$ approaches infinity for any given sample size, the finite correction factor approaches the value one. This explains why the correction factor is omitted in the formula for $\sigma_{\bar{X}}$ for infinite populations. As a working rule, the finite correction factor is often omitted when $n$ is less than five per cent of $N$.

**6.** (a) Since 10% of $\mu$ = \$4 thousand is .4, it is desired to find $P(3.6 \leqslant \bar{X} \leqslant 4.4)$. $z = (3.6 - 4)/.58 = -.69$ and $z = (4.4 - 4)/.58 = +.69$. Thus we have:

$$P(3.6 \leqslant \bar{X} \leqslant 4.4) = P(-.69 \leqslant z \leqslant +.69) = .2549 + .2549 = .5098.$$

(b) For $n = 100$, $\sigma_{\bar{X}} = \sqrt{\frac{5000-100}{5000-1}} \cdot \frac{12}{\sqrt{100}} = 1.19$;

$z = (3 - 4)/1.19 = -.84$ and $z = (5 - 4)/1.19 = +.84$;

$P(3 \leqslant \bar{X} \leqslant 5) = P(-.84 \leqslant z \leqslant +.84) = .2995 + .2995 = .5990$.

## REVIEW PROBLEMS

1. In the past, the mean claim size for a given fire insurance policy has been \$1249, and the standard deviation of the claim sizes has been \$1196.
   (a) There is a backlog of 676 claims to be processed. Assuming these claims represent a simple random sample from the infinite population associated with the claims process, with $\mu$ = \$1249 and $\sigma$ = \$1196, what is the probability that the mean claim size $\bar{X}$ in the backlog exceeds \$1350? _____
   (b) If the actual mean claim size for these 676 claims turns out to be $\bar{X}$ = \$1486, is there reason to believe that these claims are not a simple random sample from a population with $\mu$ = \$1249 and $\sigma$ = \$1196?
   _____ √

2. Acme Used Car Sales wishes to estimate the mean number of days that elapse between the time a car is put on the lot and the time it is sold from a simple random sample of sales records. Two sample sizes are proposed, $n = 64$ and $n = 100$.
   (a) Is there any guarantee that one sample size will yield an $\bar{X}$ value closer to the true mean than would be obtained with the other sample size? _____
   _____
   (b) The precision of $\bar{X}$ as an estimator of $\mu$ depends on the variability of the sampling distribution, that is, on $\sigma_{\bar{X}}$. Although $\sigma_{\bar{X}}$ is not known in this problem, can sample information about it be obtained?
   _____ √

6.(cont'd.)

(c) Decreasing $n$ from 400 to 100 produces a decrease in $P(3 \leqslant \bar{X} \leqslant 5)$ from .9146 to .5990.

(d) For $\sigma = 24$, $\sigma_{\bar{X}} = \sqrt{\frac{5000 - 400}{5000 - 1}} \cdot \frac{24}{\sqrt{400}} = 1.15$;

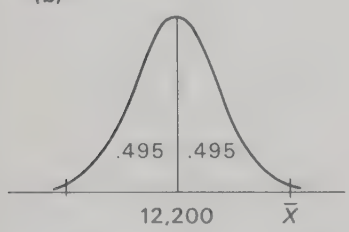
$z = (3 - 4)/1.15 = -.87$ and $z = (5 - 4)/1.15 = +.87$;

$P(3 \leqslant \bar{X} \leqslant 5) = P(-.87 \leqslant z \leqslant +.87) = .3078 + .3078 = .6156$.

(e) An increase in $\sigma$ from 12 to 24 produces a decrease in $P(3 \leqslant \bar{X} \leqslant 5)$ from .9146 to .6156. Comparing this effect with that noted in (c), we see that doubling the magnitude of $\sigma$ is approximately equivalent to reducing $n$ to 25% of its original value in terms of the effect on a probability statement such as $P(3 \leqslant \bar{X} \leqslant 5)$.

7. (a) $\mu_{\bar{X}} = \mu = \$12{,}200$; $\sigma_{\bar{X}} = \sigma/\sqrt{n} = 1400/\sqrt{49} = \$200$.

(b)




The $z$ value corresponding to an area of .495 between the mean and the upper limit of $\bar{X}$ is +2.58. Hence: $+2.58 = (\bar{X} - 12{,}200)/200$. Solving gives: $\bar{X} = 12{,}716$; by symmetry, the lower limit is 11,684. Thus $\bar{X}$ will fall in the range \$11,684 to \$12,716, 99% of the time.

(c) $\sigma_{\bar{X}} = \sigma/\sqrt{n} = 100$. Since $\sigma = 1400$, we must solve $1400/\sqrt{n} = 100$ for $n$. Solving gives $n = 196$.

(d) Exactly normal with $\mu_{\bar{X}} = \$12{,}200$ and $\sigma_{\bar{X}} = 1400/\sqrt{4} = \$700$.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $\mu_{\bar{X}} = 1249$; $\sigma_{\bar{X}} = 1196/\sqrt{676} = 46$;

$z = (1350 - 1249)/46 = +2.20$; $P(\bar{X} > 1350) = P(z > +2.20) = .0139$.

(b) If $\mu = 1249$ and $\sigma = 1196$, the probability of obtaining an $\bar{X}$ of \$1486 or more is exceedingly small. This would suggest that the claims do not represent a simple random sample from the indicated population.

2. (a) There is no guarantee that any one sample size will lead to an $\bar{X}$ closer to $\mu$ than any other sample size. However, the larger the sample size, the greater the probability that $\bar{X}$ will fall in a given range about $\mu$, other things remaining the same.

(b) Since $\sigma_{\bar{X}} = \sigma/\sqrt{n}$, knowledge about the value of $\sigma$ can be used to assess the value of $\sigma_{\bar{X}}$. Knowledge of $\sigma$ is obtained by using the sample standard deviation, $s$.

# 10

# Sampling Distribution of $\bar{p}$

The population proportion $p$ is estimated by the sample proportion $\bar{p}$. We now consider the sampling distribution of $\bar{p}$.

**A. Finite Population (Sampling Without Replacement).**

Example 1 A population of six households is to be sampled to determine the proportion of households that use gas water heaters. A simple random sample of two households is to be selected from this population. The population and the derivation of the sampling distribution of $\bar{p}$ are shown below:

**Table 10.1**

| Population | |
|---|---|
| Household | Type of Water Heater |
| *A* | Not Gas |
| *B* | Gas |
| *C* | Gas |
| *D* | Gas |
| *E* | Not Gas |
| *F* | Gas |

| Sampling Distribution of $\bar{p}$ | |
|---|---|
| $\bar{p}$ | $P(\bar{p})$ |
| 0 | 1/15 |
| 1/2 | 8/15 |
| 1 | 6/15 |
| | 1.0 |

| Samples | | |
|---|---|---|
| Households | $\bar{p}$ | Probability |
| *A, B* | 1/2 | 1/15 |
| *A, C* | 1/2 | 1/15 |
| *A, D* | 1/2 | 1/15 |
| *A, E* | 0 | 1/15 |
| *A, F* | 1/2 | 1/15 |
| *B, C* | 1 | 1/15 |
| *B, D* | 1 | 1/15 |
| *B, E* | 1/2 | 1/15 |
| *B, F* | 1 | 1/15 |
| *C, D* | 1 | 1/15 |
| *C, E* | 1/2 | 1/15 |
| *C, F* | 1 | 1/15 |
| *D, E* | 1/2 | 1/15 |
| *D, F* | 1 | 1/15 |
| *E, F* | 1/2 | 1/15 |

**Problem 1**

Refer to Example 1.

(a) What is the value of the population proportion $p$? _____

(b) What is the probability that $\bar{p}$ will be within ± .20 of $p$? _____

(c) Is the sampling distribution of $\bar{p}$ here skewed or symmetrical? _____ √

**Answers**

1. (a) $p = 2/3$.
   (b) $P(.47 \leqslant \bar{p} \leqslant .87) = 8/15 = .53$. *Note*: $p + .20 = .67 + .20 = .87$;
   $p - .20 = .67 - .20 = .47$.
   (c) The sampling distribution of $\bar{p}$ is skewed for all $p$ values except $p = .5$. Here $p = 2/3$, thus the sampling distribution is skewed.

**Problem 2**

(a) For the population in Example 1, obtain the sampling distribution of $\bar{p}$ for a simple random sample of size four. √

(b) For the sampling distribution of $\bar{p}$ obtained in (a) above, what is the probability that $\bar{p}$ will be within $\pm.20$ of $p$? _____

(c) Comparing the results from Problem 1(b) and 2(b), what is the effect of increasing $n$? _____
_____ √

**B. Mean and Standard Deviation of Sampling Distribution of $\bar{p}$.** Let $\mu_{\bar{p}}$ and $\sigma_{\bar{p}}$ denote the mean and standard deviation respectively of the sampling distribution of $\bar{p}$. For a finite population and sampling without replacement, we have:

$$\mu_{\bar{p}} = p \quad \text{and} \quad \sigma_{\bar{p}} = \sqrt{\frac{N-n}{N-1}} \cdot \sqrt{\frac{p(1-p)}{n}} .$$

Here $p$ is the population proportion, $N$ is the population size, and $n$ is the sample size. The multiplier $\sqrt{(N-n)/(N-1)}$ is called the *finite correction factor.*

Example 2 For the problem described in Example 1, $p = 2/3$, $N = 6$, and $n = 2$. Hence the sampling distribution of $\bar{p}$ has the following mean and standard deviation:

$$\mu_{\bar{p}} = p = 2/3 \quad \text{and} \quad \sigma_{\bar{p}} = \sqrt{\frac{6-2}{6-1}} \cdot \sqrt{\frac{(2/3)(1/3)}{2}} = .30.$$

These values for $\mu_{\bar{p}}$ and $\sigma_{\bar{p}}$ also could have been obtained from the sampling distribution of $\bar{p}$ by direct computation.

**Problem 3**

Refer to Problem 2. Calculate $\mu_{\bar{p}}$ and $\sigma_{\bar{p}}$ for the sampling distribution of $\bar{p}$ in that problem, using the formulas given above. $\mu_{\bar{p}} =$_____ ; $\sigma_{\bar{p}} =$ _____ .√

**Problem 4**

(a) Is $\bar{p}$ an unbiased estimator of $p$? _____

(b) Compare $\sigma_{\bar{p}}$ for $n = 2$ and $n = 4$ (Example 2 and Problem 3). How does the precision of $\bar{p}$, as an estimator of $p$, change with an increase in the sample size? _____

_____ √

**C. Infinite Population (or Finite Population — Sampling With Replacement).** For infinite populations (or for finite populations if sampling is with replacement), the sampling distribution of $\bar{p}$ follows a binomial probability distribution. The binomial probability distribution, discussed in Section 7, Topic A, is described by the following formula:

$$P(\bar{p}) = P(X/n) = \binom{n}{X} p^X (1-p)^{n-X} \quad \text{for } X = 0, 1, \ldots, n.$$

Here $X$ is the number of occurrences of the characteristic under consideration in a sample of size $n$. For instance, $X$ may be the number of houses in the sample that use gas water heaters. By definition, $\bar{p} = X/n$.

The mean and standard deviation of the sampling distribution of $\bar{p}$ are given by:

$$\mu_{\bar{p}} = p \quad \text{and} \quad \sigma_{\bar{p}} = \sqrt{\frac{p(1-p)}{n}}.$$

These relations are the same as those for a finite population when sampling is without replacement, except for the omission of the finite correction factor in $\sigma_{\bar{p}}$. The finite correction factor is omitted because it approaches the value one as the population size approaches infinity.

Example 3 Ten per cent of the items produced by a process are defective. An inspector takes a simple random sample of four items and determines the proportion of those which are defective. The sampling distribution of the resulting proportion follows a binomial probability distribution with $n = 4$, and $p = .10$. The probabilities for the possible $\bar{p}$ values are obtained directly from Table A-4 and are presented below:

| $\bar{p} = X/n$ | $P(\bar{p})$ |
|---|---|
| 0 | .6561 |
| 1/4 | .2916 |
| 2/4 | .0486 |
| 3/4 | .0036 |
| 1 | .0001 |
| | 1.0000 |

**Answers**

2. (a)

| Sample | | | Sampling Distribution of $\bar{p}$ | |
|---|---|---|---|---|
| Households | $\bar{p}$ | Probability | $\bar{p}$ | $P(\bar{p})$ |
| *A, B, C, D* | 3/4 | 1/15 | 1/2 | 6/15 |
| *A, B, C, E* | 1/2 | 1/15 | 3/4 | 8/15 |
| *A, B, C, F* | 3/4 | 1/15 | 1 | 1/15 |
| *A, B, D, E* | 1/2 | 1/15 | | 1.0 |
| *A, B, D, F* | 3/4 | 1/15 | | |
| *A, B, E, F* | 1/2 | 1/15 | | |
| *A, C, D, E* | 1/2 | 1/15 | | |
| *A, C, D, F* | 3/4 | 1/15 | | |
| *A, C, E, F* | 1/2 | 1/15 | | |
| *A, D, E, F* | 1/2 | 1/15 | | |
| *B, C, D, E* | 3/4 | 1/15 | | |
| *B, C, D, F* | 1 | 1/15 | | |
| *B, C, E, F* | 3/4 | 1/15 | | |
| *B, D, E, F* | 3/4 | 1/15 | | |
| *C, D, E, F* | 3/4 | 1/15 | | |

(b) $P(.47 \leqslant \bar{p} \leqslant .87) = 6/15 + 8/15 = 14/15 = .93.$

(c) Increasing the sample size from two to four has increased the probability of $\bar{p}$ being within .20 of $p$ from .53 to .93.

3. $\mu_{\bar{p}} = p = 2/3;$

$$\sigma_{\bar{p}} = \sqrt{\frac{6-4}{6-1}} \cdot \sqrt{\frac{(2/3)(1/3)}{4}} = .15.$$

4. (a) Since $\mu_{\bar{p}} = p$, it follows by definition that $\bar{p}$ is an unbiased estimator of $p$.

(b)

| $n$ | $\sigma_{\bar{p}}$ |
|---|---|
| 2 | .30 |
| 4 | .15 |

$\sigma_{\bar{p}}$ decreases with increasing $n$. This also can be seen from the formula for $\sigma_{\bar{p}}$.

Using the above formulas for the mean and standard deviation of the sampling distribution of $\bar{p}$, the following values are obtained:

$$\mu_{\bar{p}} = p = .10 \quad \text{and} \quad \sigma_{\bar{p}} = \sqrt{(.10)(.90)/4} = .15.$$

**Problem 5**

Ten per cent of the items produced by a process are defective. An inspector selects a simple random sample of six of these items.

(a) Obtain the sampling distribution of $\bar{p}$. $\surd$

(b) Obtain the values of $\mu_{\bar{p}}$ and $\sigma_{\bar{p}}$ for this sampling distribution, using the formulas given above.
$\mu_{\bar{p}} =$_____; $\sigma_{\bar{p}} =$_____
(c) Is this sampling distribution of $\bar{p}$ symmetrical or skewed? _____ √

**D. Central Limit Theorem.** The central limit theorem is applicable to the sampling distribution of $\bar{p}$. Thus:

> *The sampling distribution of $\bar{p}$ is approximately normal if the simple random sample size is sufficiently large.*

For finite populations when sampling without replacement, the population size must be considerably larger than the sample size if this theorem is to apply.

A working rule frequently used to indicate situations in which the sampling distribution of $\bar{p}$ may be approximated by a normal distribution states that the normal approximation is appropriate if both $np$ and $n(1-p)$ are greater than five.

Example 4 Thirty per cent of the inventory items of a firm have high annual turnover rates. Suppose that the firm's management wishes to know the probability that 40% or more of the inventory items in a simple random sample of 100 items will have high turnover rates. Assume that the number of items in the firm's inventory is large so that the finite correction factor may be ignored.

To obtain the desired probability it is necessary to construct the sampling distribution of $\bar{p}$. Since both $np = 100\,(.30) = 30$ and $n(1-p) = 100(.70) = 70$ are greater than five, the central limit theorem states that the sampling distribution of $\bar{p}$ is approximately normal. We know that the mean and standard deviation of this normal approximation are $\mu_{\bar{p}} = .30$ and $\sigma_{\bar{p}} = \sqrt{p(1-p)/n} = \sqrt{(.30)(.70)/100} = .0458$. The $z$ value corresponding to .40 is $z = (.40 - .30)/.0458 = 2.18$; hence it follows that:

$$P(\bar{p} \geqslant .40) = P(z \geqslant 2.18) = .5000 - .4854 = .0146.$$

Thus, the probability is .0146 that at least 40% of the inventory items in a simple random sample of 100 items will have high turnover rates. As a matter of interest, if the binomial probability distribution had been used, rather than the normal approximation, the exact probability $P(\bar{p} \geqslant .40) = .0210$ would have been obtained.

In some books, a correction for continuity is applied when a binomial probability distribution is approximated by a normal probability distribution. That correction will not be discussed here.

**Problem 6**

A firm has prepared a mail solicitation to sell its product. It is felt that a response rate of at least 5% is required for the undertaking to be profitable. As a pilot test, 400 of the solicitations are to be sent to names selected at random from a large mailing list.

(a) What is the probability that the test will yield a response of 5% or more if the actual response rate for the population is only 4%? _____
(b) How does this probability change if the test size is increased to 900? _____ √

Answers

**5.** **(a)** $n = 6$; $p = .10$.

| $\bar{p}$ | $P(\bar{p})$ |
|---|---|
| 0 | .5314 |
| 1/6 | .3543 |
| 2/6 | .0984 |
| 3/6 | .0146 |
| 4/6 | .0012 |
| 5/6 | .0001 |
| 1 | .0000 |
| | 1.0000 |

**(b)** $\mu_{\bar{p}} = p = .10$; $\sigma_{\bar{p}} = \sqrt{(.10)(.90)/6} = .12$.

**(c)** Skewed, because $p = .10 \neq .5$.

**6.** **(a)** $n = 400$; $p = .04$; $\mu_{\bar{p}} = .04$; $\sigma_{\bar{p}} = \sqrt{(.04)(.96)/400} = .00980$.

Using the normal approximation:

$z = (.05 - .04)/.00980 = 1.02$.

Hence: $P(\bar{p} \geqslant .05) = P(z \geqslant 1.02) = .1539$.

**(b)** $n = 900$; $\sigma_{\bar{p}} = \sqrt{(.04)(.96)/900} = .00653$.

Using the normal approximation:

$z = (.05 - .04)/.00653 = 1.53$.

Hence: $P(\bar{p} \geqslant .05) = P(z \geqslant 1.53) = .0630$.

## REVIEW PROBLEMS

1. The median family income in a region is known to be \$4900, with half of the families having incomes under \$4900.
   (a) For a simple random sample of eight families from this region, construct the sampling distribution of the proportion of families in the sample with incomes less than the median income for the region. Assume the binomial probability distribution is appropriate. √

   (b) What are the mean and standard deviation of the sampling distribution in (a)? _____ _____ √

2. For an infinite population, the standard deviation of the sampling distribution of $\bar{p}$ is given by $\sigma_{\bar{p}} = \sqrt{p(1-p)/n}$.
   (a) Given $n = 100$, calculate $\sigma_{\bar{p}}$ for $p = .3, .4, .5$, and $.6$. _____ _____ _____ _____
   (b) What is the effect on $\sigma_{\bar{p}}$ of moving $p$ closer to .5? _____ _____
   (c) For what value of $p$ is $\sigma_{\bar{p}}$ a maximum? _____
   (d) Does your answer in (c) hold for other values of $n$? _____ √

3. An electric utility company is undertaking a marketing survey to determine the proportion of households, $p$, which use electric cooking ranges. What sample size is required for the survey in order to ensure that the value of $\sigma_{\bar{p}}$ does not exceed .01? Ignore the finite correction factor and assume that the utility company has no knowledge about the true value of $p$. (*Hint*: Use the value of $p$ for which $\sigma_{\bar{p}}$ is maximum.) _____ _____ √

4. In a work sampling study, a machine is observed at random points in time to determine the proportion of time it is operating. If the machine actually operates 80% of the time, what is the probability that in 50 observations it will be found to be operating 35 times or less? (*Hint*: Is the normal approximation applicable?) _____ √

5. A firm has a large number of employees. It is desired to know the proportion of these employees who have lost more than three days from work due to illness during the past year. A simple random sample of 50 employees reveals that 36% of them lost this much time from work. In order to assess the precision of the sample proportion, $\bar{p} = .36$, it is necessary to have information about the value of $\sigma_{\bar{p}}$. How can $\sigma_{\bar{p}}$ be estimated? _____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a) $n = 8$; $p = .5$.

| $\bar{p}$ | $P(\bar{p})$ |
|---|---|
| 0 | .0039 |
| 1/8 | .0312 |
| 2/8 | .1094 |
| 3/8 | .2188 |
| 4/8 | .2734 |
| 5/8 | .2188 |
| 6/8 | .1094 |
| 7/8 | .0312 |
| 1 | .0039 |
| | 1.0000 |

(b) $\mu_{\bar{p}} = p = .5$; $\sigma_{\bar{p}} = \sqrt{(.5)(.5)/8} = .177$.

2. (a)

| $p$ | $\sigma_{\bar{p}}$ |
|---|---|
| .3 | .0458 |
| .4 | .0490 |
| .5 | .0500 |
| .6 | .0490 |

(b) As $p$ approaches .5 from either direction, $\sigma_{\bar{p}}$ gets larger.

(c) $p = .5$.

(d) Yes. Since $p(1 - p)$ is a maximum when $p = .5$, it follows that $\sigma_{\bar{p}}$ is a maximum when $p = .5$ for any $n$.

3. $\sigma_{\bar{p}} = \sqrt{p(1-p)/n} \leqslant \sqrt{(.5)(.5)/n} \leqslant .01$.

The first inequality was demonstrated in Review Problem 2. The second inequality guarantees that $\sigma_{\bar{p}} \leqslant .01$. Rearranging the second inequality we see that $n \geqslant 2500$ satisfies the specification on $\sigma_{\bar{p}}$.

4. $n = 50$; $p = .80$; $\mu_{\bar{p}} = .80$; $\sigma_{\bar{p}} = \sqrt{(.80)(.20)/50} = .0566$.

Using the normal approximation:

$z = (35/50 - .80)/.0566 = -1.77$.

Hence: $P(\bar{p} \leqslant 35/50) = P(z \leqslant -1.77) = .0384$.

5. Since $\sigma_{\bar{p}} = \sqrt{p(1-p)/n}$, an estimate of $\sigma_{\bar{p}}$ may be obtained by employing the sample proportion $\bar{p}$ in place of $p$.

# 11

# Interval Estimation of the Population Mean – Large Simple Random Sample

In statistical estimation, population characteristics are estimated from sample data. In this section, we are concerned with estimating the population mean using data from a large simple random sample. The type of estimate to be used is an interval estimate.

**A. Confidence Interval.** The following interval estimate of the population mean, $\mu$, is called a confidence interval:

$$\bar{X} - z\,\sigma_{\bar{X}} \leqslant \mu \leqslant \bar{X} + z\,\sigma_{\bar{X}}.$$

Here $\bar{X}$ is the mean of the sample which is being used to estimate $\mu$, $z$ is a specified value of the standard normal deviate, and $\sigma_{\bar{X}}$ is the standard deviation of the sampling distribution of $\bar{X}$, for brevity usually called the standard deviation of $\bar{X}$. Recall that $\sigma_{\bar{X}} = \sigma/\sqrt{n}$ for infinite populations and $\sigma_{\bar{X}} = \sqrt{(N-n)/(N-1)} \cdot \sigma/\sqrt{n}$ for finite populations.

In most problems, the value of $\sigma$ and hence the value of $\sigma_{\bar{X}}$ are not known. However, we may estimate $\sigma$ by the sample standard deviation $s$. We then obtain:

$$s_{\bar{X}} = s/\sqrt{n} \text{ or } s_{\bar{X}} = \sqrt{(N-n)/(N-1)} \cdot s/\sqrt{n},$$

as the case may be, where $s_{\bar{X}}$ denotes the estimator of $\sigma_{\bar{X}}$. Substituting $s_{\bar{X}}$ for $\sigma_{\bar{X}}$ in the confidence interval formula gives:

$$\bar{X} - zs_{\bar{X}} \leqslant \mu \leqslant \bar{X} + zs_{\bar{X}}.$$

When $n$ is large, as is assumed here, no significant effect is introduced by using $s_{\bar{X}}$ instead of $\sigma_{\bar{X}}$ in the confidence interval formula.

Example 1 It is desired to estimate the mean time required to perform a task. A simple random sample of 49 time measurements is obtained. The sample results are $\bar{X} = 13.0$ minutes and $s = 5.6$ minutes.

The estimate of $\sigma_{\bar{X}}$ is $s_{\bar{X}} = 5.6/\sqrt{49} = .8$. If a $z$ value of 3 is specified, the interval estimate for $\mu$ is:

$$13.0 - (3)(.8) \leqslant \mu \leqslant 13.0 + (3)(.8), \quad \text{which gives} \quad 10.6 \leqslant \mu \leqslant 15.4.$$

Thus, we estimate that the mean time is between 10.6 and 15.4 minutes. (The role of the $z$ value is discussed under Topic B.) The limits 10.6 and 15.4 are called *confidence limits*.

**Problem 1**

A television station wishes to estimate the mean TV viewing time of families on Saturday evenings. To obtain this estimate, 64 families were selected at random. The sample results are $\bar{X} = 1.5$ hours and $s = .8$ hours.

(a) Construct a confidence interval for the mean viewing time. Use a $z$ value of 2. _____

(b) If a second sample of 64 families were selected, would these same confidence limits be obtained? _____ Explain. _____ √

**B. Confidence Coefficient.** The confidence coefficient assigned to a confidence interval indicates the probability that the procedure employed will furnish confidence limits which include the population parameter of interest. The confidence coefficient is related to the $z$ value used in setting up the confidence interval.

Example 2 In Example 1, a $z$ value of 3 was used in constructing the confidence interval. Since $P(-3 \leqslant z \leqslant +3) = .997$, it follows by definition that the confidence coefficient in Example 1 is 99.7%. This means that if we took many independent samples of size 49 and constructed an interval estimate for each, using $z = 3$, about 99.7% of these estimates would be correct, that is, would include the true value of $\mu$ within the limits.

**Problem 2**

(a) What is the confidence coefficient of the confidence interval constructed in Problem 1(a)? _____

(b) What $z$ value should have been used in Problem 1(a) if a confidence coefficient of 90% were desired? _____

(c) For given sample results, what is the effect on the width of the confidence interval of reducing $z$? _____

(d) For given sample results, what is the effect on the confidence coefficient of reducing $z$? _____ √

**C. Determination of Necessary Sample Size.** Suppose that the desired confidence interval is to have a *half-width* $h$ for a specified confidence coefficient, that is, the confidence limits are to be $\bar{X} \pm h$. The specification of $h$ and the confidence coefficient (in terms of $z$), plus an advance judgment of $\sigma$, permits the determination of the necessary sample size:

$$n = \frac{z^2 \sigma^2}{h^2} \quad \text{for infinite populations;}$$

$$n = \frac{\sigma^2}{\frac{h^2}{z^2} + \frac{\sigma^2}{N}} \quad \text{for finite populations } (N \text{ is population size}).$$

Example 3 The quality of an incoming shipment of pieces depends on the mean number of defects per piece in the shipment. It is desired to obtain a confidence interval for this mean with a half-width of .2 at the 95.4% confidence level. From past experience with shipments from this same supplier, $\sigma$ is taken to be approximately 1.2. Assume the population size is very large so that it can be treated as an infinite population.

We have: $\sigma = 1.2$, $z = 2$, and $h = .2$. Thus the required sample size is:

$$n = \frac{(2)^2(1.2)^2}{(.2)^2} = 144.$$

**Problem 3**

Refer to Example 3.

(a) Suppose the random sample of 144 items was inspected and the sample results were $\bar{X} = 1.5$ and $s = 1.6$. What is the confidence interval for $\mu$? _____ √

(b) Does the confidence interval provide the desired precision? _____

(c) What, if anything, "went wrong" in the advance planning? _____

(d) On the basis of the confidence interval in (a), what conclusion would you draw about how the quality of this shipment compares with past shipments in which there have been 1.0 defects per piece?

_____ √

**Problem 4**

A study of households is to be undertaken to estimate the mean annual expenditures on residential home improvements. It is desired to estimate this mean within ± \$10 with a confidence coefficient of 99.7%. It is estimated that $\sigma$ = \$200.

(a) What sample size should be used in the study? (Assume the population is very large.) _____ √

(b) If a 95.4% confidence coefficient were acceptable, what would the necessary sample size be? _____

(c) Does increasing the confidence coefficient from 95.4% to 99.7% have a substantial effect on the needed sample size? _____

_____ √

Answers

1. (a) $s_{\bar{X}} = s/\sqrt{n} = .8/\sqrt{64} = .1$;

$$1.5 - 2(.1) \leqslant \mu \leqslant 1.5 + 2(.1), \text{ or}$$
$$1.3 \leqslant \mu \leqslant 1.7.$$

(b) Not likely. Different samples generally yield different values for $\bar{X}$ and $s$, both of which affect the confidence limits.

2. (a) $z = 2$; confidence coefficient = $P(-2 \leqslant z \leqslant +2) \times 100 = 95.4\%$.
(b) $z = 1.64$.
(c) For given sample results, the confidence interval is shortened by reducing $z$.
(d) For given sample results, reducing $z$ decreases the confidence coefficient.

3. (a) For $n = 144$, $\bar{X} = 1.5$, $s = 1.6$, and a confidence coefficient of 95.4%, the confidence interval equals:

$$1.5 - 2(.133) \leqslant \mu \leqslant 1.5 + 2(.133) \quad \text{or} \quad 1.234 \leqslant \mu \leqslant 1.766,$$
$$\text{where } s_{\bar{X}} = 1.6/\sqrt{144} = .133.$$

(b) **No. The half-width of .266 exceeds the desired half-width of .2.**
(c) The sample standard deviation $s = 1.6$ suggests that the planning value $\sigma = 1.2$ was an underestimate of the true value of $\sigma$. Nevertheless, the confidence interval obtained in (a) is valid, and may still be quite satisfactory, the width being of roughly the same order of magnitude as that desired. If the confidence interval were too wide to be useful, a larger sample would be required.
(d) **The fact that the confidence interval in (a) is entirely above 1.0 is evidence that this shipment is of poorer quality than past shipments.**

4. (a) $z = 3$, $h = 10$, $\sigma = 200$. Thus $n = \dfrac{(3)^2(200)^2}{(10)^2} = 3600$.

(b) $z = 2$, $h = 10$, $\sigma = 200$. Thus $n = \dfrac{(2)^2(200)^2}{(10)^2} = 1600$.

(c) **Increasing the confidence coefficient by 4.3% (from 95.4% to 99.7%) has more than doubled the required sample size.**

## REVIEW PROBLEMS

1. A firm wishes to test the effectiveness of a new apprenticeship training program. Seventy-two new apprentices were given an aptitude test and then grouped into 36 pairs, with the apprentices in each pair having about the same aptitude score. One of the apprentices in each pair was randomly selected for the new training program, the other apprentice for the old program. After completing the training programs, all trainees wrote a journeyman examination. For each pair, let $X$ = the grade for the apprentice who took the new training program minus the grade for the apprentice who took the old program. The sample results are $\overline{X} = 4$ and $s = 10$.
   (a) What is $\overline{X}$ an estimate of? _____ √
   (b) What is the relevant sample size for calculating $s_{\overline{X}}$? _____ √
   (c) Construct a 95.4% confidence interval for the mean difference in the grades on the journeyman examination. _____
   (d) Is there reason to believe that the new training program is effective in increasing the mean grade of apprentices on the journeyman examination? _____
   (e) In terms of the design of the study, what advantage was gained by pairing apprentices who had approximately the same scores on the aptitude test? _____ √

2. An analyst wishes to estimate the mean amount of total assets which are financed by long term debt in a population consisting of 500 companies. He requires a confidence interval with a half-width of $50,000 at the 90% confidence level. The analyst is uncertain about the specific value of $\sigma$, but reasons that it must be somewhere between $100,000 and $500,000.
   (a) The analyst would rather have a sample which is too large than too small. What value of $\sigma$ should he use in his sample size formula? _____ √
   (b) Calculate the sample size that he should use. _____
   (c) The analyst feels the sample size obtained in (b) is too large considering the time he has available for the study. What would be the effect of reducing the sample size? _____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a) $\bar{X}$ is always an estimate of the population mean $\mu$. In this case, $\mu$ refers to differences in the grades for the two apprentices in a pair. More specifically, if $\mu_1$ and $\mu_2$ are the mean grades for apprentices who have been trained under the old and new programs respectively, then the population mean $\mu$, which is being estimated, is the difference $\mu_2 - \mu_1$.

   (b) 36.

   (c) $s_{\bar{X}} = 10/\sqrt{36} = 1.67$;
   $4 - 2(1.67) \leqslant \mu \leqslant 4 + 2(1.67)$, or
   $.67 \leqslant \mu \leqslant 7.33$.

   (d) **The fact that the confidence interval in (c) is entirely above zero is evidence that the new training program leads to higher grades on the average than the old program. The strength of this evidence is measured by the confidence coefficient, 95.4% in this case.**

   (e) **By pairing apprentices with approximately the same aptitude scores, the aptitude variable is partially controlled. The pairing prevents any grade differences which result from different aptitudes from being mixed with the grade differences which result from different training programs. As a result, the sampling error in our estimate of $\mu$ is reduced.**

2. (a) $\sigma = \$500{,}000$.

   (b) $$n = \frac{(500{,}000)^2}{\frac{(50{,}000)^2}{(1.64)^2} + \frac{(500{,}000)^2}{500}} = 269.$$

   (c) **Reducing the sample size will reduce the precision of his estimate. Note however, that if the actual value of $\sigma$ is less than the planning value of $500,000 then a reduced sample size may still provide him with an estimate of $\mu$ with the desired precision.**

# 12

# Interval Estimation of the Difference Between Two Population Means – Large Simple Random Samples

In this section, we take up an interval estimate of the difference between two population means for the case where the random sample from each population is large, and the two samples are selected independently.

**A. Confidence Interval.** The two populations under study have means and standard deviations of $\mu_1$ and $\mu_2$, and $\sigma_1$ and $\sigma_2$, respectively. The sizes of the samples from the two populations are $n_1$ and $n_2$ respectively. The two samples are statistically independent.

The confidence interval for the difference between the two population means, $\mu_2 - \mu_1$, is:

$$D - z\sigma_D \leqslant \mu_2 - \mu_1 \leqslant D + z\sigma_D.$$

Here $D = \bar{X}_2 - \bar{X}_1$ where $\bar{X}_2$ and $\bar{X}_1$ are the means of the two samples. Using earlier probability theorems, we know

$$E(D) = E(\bar{X}_2 - \bar{X}_1) = \mu_2 - \mu_1, \quad \text{and} \quad \sigma_D = \sqrt{\sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2}}$$

because of the independence of the two samples. If the population is infinite or the finite correction factor can be ignored, we have:

$$\sigma_D = \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}.$$

In most problems, the values of $\sigma_1$ and $\sigma_2$ are not known. However, since $n_1$ and $n_2$ are large, no significant effect is introduced by using the sample standard deviations, $s_1$ and $s_2$, in place of $\sigma_1$ and $\sigma_2$ respectively in the confidence interval formula. Letting $s_D = \sqrt{s^2_{\bar{X}_1} + s^2_{\bar{X}_2}}$, the confidence interval becomes:

$$D - zs_D \leqslant \mu_2 - \mu_1 \leqslant D + zs_D.$$

Example 1 In March, 1968, a simple random sample of 1000 house renters was selected to estimate the mean rent paid by the population of renters during that month. In March, 1969, another simple random sample of the same size was selected for the same purpose, independently of the first sample. The sample results were as follows:

| 1968 Sample | 1969 Sample |
|---|---|
| $\bar{X}_1$ = \$120 | $\bar{X}_2$ = \$131 |
| $s_1$ = \$ 20 | $s_2$ = \$ 25 |
| $n_1$ = 1000 | $n_2$ = 1000 |

The population of house renters in each year was very large.

Suppose that a 95.4% confidence interval for the difference in the mean rent paid in the two years is desired. We have $D = 131 - 120 = \$11$, $z = 2$, and $s_D = \sqrt{(20)^2/1000 + (25)^2/1000} = \$1.01$. Hence the confidence interval is:

$11 - (2)(1.01) \leq \mu_2 - \mu_1 \leq 11 + (2)(1.01)$ which gives $\$8.98 \leq \mu_2 - \mu_1 \leq \$13.02$. Thus, the mean rent paid by renters increased between $8.98 and $13.02 from March 1968 to March 1969. The confidence we have in the correctness of this estimate is 95.4%.

**Problem 1**

Refer to Example 1. If the same sample of renters had been used in 1969 as in 1968, would the statistical technique used in Example 1 have been appropriate? Explain. _____

_____

_____ √

**Problem 2**

An educator investigated the effect of the use of a programmed learning book on the performance of students. The experiment involved two classes, containing 49 students and 36 students respectively. During the academic year, the first class used the programmed learning book and the second class used the standard textbook. The results of the final examination are shown below:

| Class that Used the Programmed Learning Book | Class that Used the Standard Textbook |
|---|---|
| $\bar{X}_1 = 68.3$ | $\bar{X}_2 = 64.8$ |
| $s_1 = 14.7$ | $s_2 = 18.0$ |
| $n_1 = 49$ | $n_2 = 36$ |

Assume that the classes represent independent, simple random samples of students from infinite populations.

(a) Construct the 99% confidence interval for the difference between the population mean scores of students using the programmed learning book and students using the standard textbook. _____

_____ √

(b) From the confidence interval in (a), is there strong evidence that the use of the programmed learning book changes the mean score of students on the final examination as compared with the standard textbook?

_____ √

## REVIEW PROBLEMS

1. To test the effectiveness of a new apprenticeship training program, 72 new apprentices are randomly divided into two groups of 36. The first group undergoes training under the new program, while the second group undergoes training under the old one. At the end of the training programs, each apprentice writes a journeyman examination. Suppose the results are as follows:

| New Training Program | Old Training Program |
|---|---|
| $\bar{X}_1 = 76.0$ | $\bar{X}_2 = 72.0$ |
| $s_1 = 12.0$ | $s_2 = 15.0$ |
| $n_1 = 36$ | $n_2 = 36$ |

(a) Construct the 90% confidence interval for $\mu_1 - \mu_2$. (Note: $\mu_1 - \mu_2$ is to be estimated.)

(b) Interpret the meaning of the confidence interval constructed in (a).

(c) Is the new training program "better" than the old training program?

(d) Refer to Review Problem 1, Section 11. The problem situation described there differs from the above in that the apprentices were first paired on the basis of ability test scores before random assignments to the type of program were made. Would the statistical technique used above for estimating the difference in mean grades be appropriate in Review Problem 1, Section 11? Why or why not?

Answers

1. If the same sample of renters had been used, the statistical technique of Example 1 would not have been appropriate because the two samples would not be statistically independent. In that case, the technique discussed in Review Problem 1, Section 11 would be the appropriate one.

2. (a) $D = 64.8 - 68.3 = -3.5;$
$s_D = \sqrt{(14.7)^2/49 + (18.0)^2/36} = \sqrt{13.41} = 3.66;$
$z = 2.58$ for a 99% confidence coefficient.
$-3.5 - 2.58(3.66) \leqslant \mu_2 - \mu_1 \leqslant -3.5 + 2.58(3.66)$ or $-12.94 \leqslant \mu_2 - \mu_1 \leqslant 5.94.$

   (b) Since the confidence interval in (a) contains zero, at the 99% confidence level, there is no strong evidence for concluding that $\mu_1$ and $\mu_2$ differ.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $D = 76.0 - 72.0 = 4.0;$
$s_D = \sqrt{(12.0)^2/36 + (15.0)^2/36} = \sqrt{10.25} = 3.20;$
$z = 1.64$ for a 90% confidence coefficient.
$4.0 - 1.64(3.20) \leqslant \mu_1 - \mu_2 \leqslant 4.0 + 1.64(3.20)$ or $-1.25 \leqslant \mu_1 - \mu_2 \leqslant 9.25.$

   (b) The new training program changes the mean grade on the journeyman examination by an amount between −1.25 and +9.25 (90% confidence coefficient). The confidence coefficient indicates that a procedure has been used which leads to correct interval estimates 90% of the time.

   (c) The term "better" should be clarified. If the change in the mean grade is the measure of effectiveness of the new training program, then the fact that the confidence interval obtained in (a) spans zero indicates that, at the 90% confidence level, no strong evidence exists that the new training program is better.

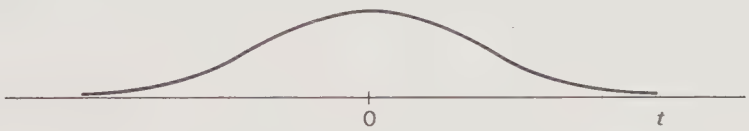
   (d) No. The statistical technique described in this section is based on statistically independent samples. Pairing the apprentices on the basis of ability test scores first, produces matched (dependent) samples. The procedure described in Review Problem 1, Section 11 should be used to construct a confidence interval for $\mu_1 - \mu_2$ when the sample observations are purposely paired.

# 13

# Interval Estimation of the Population Mean – Normal Population, Small Sample

In this section, we discuss the construction of a confidence interval for the mean of a normal population.

**A. Student's *t* distribution.** A special probability distribution which will be used shortly is called Student's *t* distribution, or simply, the *t* distribution. The *t* distribution is bell-shaped and symmetrical, of the following form:




The *t* distribution has one parameter, called degrees of freedom. The mean of the *t* distribution is zero and the variance, for degrees of freedom exceeding two, is $m/(m-2)$, where $m$ equals the number of degrees of freedom.

**Problem 1**

We wish to compare the *t* distribution with the standard normal distribution.

(a) How does the mean of *t* compare with the mean of *z*? _____

(b) What is the variance of *t* when $m = 10$? _____

(c) How does this variance compare with that for *z*? _____

(d) What is the variance of *t* when $m = 20$? _____

(e) What happens to the variance of *t* as *m* gets larger? _____

_____

(f) For large *m*, how do the mean and variance of *t* compare with the mean and variance of *z*?

_____

_____ √

Example 1 In our use of the *t* distribution, we shall require information about areas under the *t* curve. There is a different *t* distribution for each number of degrees of freedom. A summary of the essential information about these distributions is presented in Table A-5.

For the *t* distribution with five degrees of freedom, it is desired to find the *t* values for which the central area in the distribution is .95, or the total area in both tails of the *t* distribution is equal to .05. We refer in Table A-5 to the row corresponding to five degrees of freedom and the column marked .05, and find 2.571. Thus, for a *t* distribution with five degrees of freedom, the probability of obtaining a *t* value less than −2.571 or more than +2.571 is .05, or the probability is .95 of obtaining a *t* value within these limits.

**Answers**

1. (a) Both $t$ and $z$ have means of zero.
   (b) Variance of $t$ ($m = 10$) $= 10/(10 - 2) = 1.25$.
   (c) Variance of $z = 1$.
   (d) Variance of $t$ ($m = 20$) $= 20/(20 - 2) = 1.11$.
   (e) The variance of $t$ approaches one as $m$ gets larger.
   (f) $t$ and $z$ have the same mean and the variance of $t$ approaches the variance of $z$ as $m$ gets larger. In fact, the distribution of $t$ approaches the distribution of $z$ (the standard normal distribution) as $m$ gets larger.

**Problem 2**

(a) Find the $t$ value for the total tail area .01 when the number of degrees of freedom is (i) 10: $t =$_____; (ii) 20: $t =$ _____; (iii) 40: $t =$ _____ .√

(b) Compare the $t$ values obtained in (a) with the $z$ value which corresponds to a total tail area of .01 in the standard normal distribution. What do you observe? _____

(c) Find the $t$ value corresponding to a central area of .90, for 10 degrees of freedom. _____

(d) Interpolating in Table A-5, find the approximate total tail area corresponding to a $t$ value of 2.288 and 15 degrees of freedom. _____

(e) Find the area in the right-hand tail corresponding to a $t$ value of 4.303 and 2 degrees of freedom. _____√

**B. Confidence Interval For Population Mean.**

1. *Population standard deviation known.* Consider the situation where the *population is normal* and $\sigma$ is known. Then we know from earlier work that the confidence interval for $\mu$ for a simple random sample of any size is given by:

$$\bar{X} - z\sigma_{\bar{X}} \leqslant \mu \leqslant \bar{X} + z\sigma_{\bar{X}}.$$

2. *Population standard deviation unknown — large sample.* If $\sigma$ is unknown, it must be estimated from the sample standard deviation $s$, and $s_{\bar{X}}$ is used as an estimator of $\sigma_{\bar{X}}$. We noted earlier that if the sample size is large, $s_{\bar{X}}$ may be substituted directly for $\sigma_{\bar{X}}$ in the above confidence interval to obtain the following interval:

$$\bar{X} - zs_{\bar{X}} \leqslant \mu \leqslant \bar{X} + zs_{\bar{X}}.$$

Substituting $s_{\bar{X}}$ for $\sigma_{\bar{X}}$ in the formula for the confidence interval has no significant effect on the confidence coefficient when the sample size is large.

3. *Population standard deviation unknown — small sample.* When the sample size is not large and $s_{\bar{X}}$ is substituted for $\sigma_{\bar{X}}$, the confidence coefficient (i.e., the probability of a correct interval estimate) is affected. In order to be able to obtain a specified confidence coefficient, it is necessary to replace $z$ in the confidence interval by a $t$ value. This is a consequence of the fact that when the population is normal, $(\bar{X} - \mu)/s_{\bar{X}}$ follows the $t$ distribution. Hence, the confidence interval for the population mean has the following form:

$$\bar{X} - ts_{\bar{X}} \leqslant \mu \leqslant \bar{X} + ts_{\bar{X}}.$$

The determination of the $t$ value will be covered in the following example.

Example 2 It is desired to obtain a 99% confidence interval for the mean time required to perform a task. It is known that the time required follows a normal distribution. A simple random sample of sixteen time measurements yields the following results: $\bar{X} = 13.0$ minutes and $s = 5.6$ minutes.

From the sample results, we have that $s_{\bar{X}} = s/\sqrt{n} = 5.6/\sqrt{16} = 1.4$ minutes. A 99% confidence coefficient corresponds to an area of .01 in both tails combined. The relevant $t$ distribution is the one with $n - 1 = 15$ degrees of freedom. Hence, the required $t$ value corresponds to a total tail area of .01 in the $t$ distribution with 15 degrees of freedom; it is 2.947. Therefore the confidence interval is:

$$13.0 - 2.947(1.4) \leqslant \mu \leqslant 13.0 + 2.947(1.4), \quad \text{which gives} \quad 8.87 \leqslant \mu \leqslant 17.13.$$

This interval is interpreted in the usual manner.

If a confidence coefficient of 95% had been specified, the $t$ value would be that for 15 degrees of freedom and a total tail area of .05, namely $t = 2.131$, and similarly for other confidence coefficients.

**Problem 3**

An industrial caterer had the following daily sales revenues on nine different days: \$58, 42, 71, 59, 66, 49, 68, 63, 55. Assume that these revenue amounts represent a simple random sample from a normal population.

(a) Construct a 95% confidence interval for the mean daily sales revenue for the caterer. For the sample data shown above, $\bar{X} = \$59$ and $s = \$9.33$. _____ √

(b) What would the $t$ value have been if $n = 350$? _____

(c) What is the effect on the $t$ value of increasing the sample size? _____ √

**C. Confidence Interval For Difference Between Two Population Means – Equal Variances.** A special application of the $t$ distribution involves the confidence interval for the difference between two population means, $\mu_2 - \mu_1$, when both populations are normal and both have the same variance, that is, $\sigma_1^2 = \sigma_2^2 = \sigma^2$.

If $\sigma^2$ is known, then the confidence interval presented in Topic A, Section 12 is appropriate. This confidence interval is:

$$D - z\sigma_D \leqslant \mu_2 - \mu_1 \leqslant D + z\sigma_D.$$

Here $D = \bar{X}_2 - \bar{X}_1$ where $\bar{X}_2$ and $\bar{X}_1$ are the means of the two samples, and

$$\sigma_D = \sqrt{\sigma^2/n_1 + \sigma^2/n_2} = \sqrt{\sigma^2\,(1/n_1 + 1/n_2)}$$

where $n_1$ and $n_2$ are the two sample sizes.

When $\sigma^2$ is unknown, an estimate of it may be obtained from a weighted average of the two sample variances, $s_1^2$ and $s_2^2$. The appropriate weights are $n_1 - 1$ and $n_2 - 1$ respectively. The estimate obtained is called a *pooled estimate* and is denoted by $s^2$:

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}.$$

Using the pooled estimate in place of $\sigma^2$ in the formula for $\sigma_D$ gives $s_D = \sqrt{s^2(1/n_1 + 1/n_2)}$. In this case, the $t$ distribution is used since $[D - (\mu_2 - \mu_1)]/s_D$ follows the $t$ distribution. The confidence interval for $\mu_2 - \mu_1$ therefore is:

$$D - ts_D \leqslant \mu_2 - \mu_1 \leqslant D + ts_D.$$

The $t$ value in the confidence interval is obtained from the $t$ distribution with $n_1 + n_2 - 2$ degrees of freedom.

**Answers**

2. **(a) (i) 3.169; (ii) 2.845; (iii) 2.704.**
   (b) The $z$ value corresponding to a total tail area of .01 is 2.576. As the number of degrees of freedom increases, the value of $t$ for a total tail area of .01 approaches 2.576, the corresponding $z$ value.
   **(c) 1.812.**
   **(d) .04.**
   **(e) .025.**

3. **(a)** $s_{\bar{X}} = s/\sqrt{n} = 9.33/\sqrt{9} = 3.11$;
   $t = 2.306$ for 8 degrees of freedom and a total tail area of .05;
   $$59 - 2.306(3.11) \leqslant \mu \leqslant 59 + 2.306(3.11) \quad \text{or} \quad 51.83 \leqslant \mu \leqslant 66.17$$
   **(b) Approximately 1.960.**
   **(c)** The $t$ value is reduced, and approaches the corresponding $z$ value.

Example 3 A manufacturer wishes to compare the mean service life of two different types of special light bulbs, with a 90% confidence coefficient. Because a test of service life destroys the bulb, only a small simple random sample of each type of bulb is used. Suppose the following sample results are obtained:

| Type A | | | Type B | | |
|---|---|---|---|---|---|
| $\bar{X}_1 = 985$ hours | $s_1 = 189$ hours | $n_1 = 4$ | $\bar{X}_2 = 1243$ hours | $s_2 = 152$ hours | $n_2 = 6$ |

Assume that the samples are from normal populations with equal variances.

The pooled estimate of the population variance is:

$$s^2 = \frac{(4-1)(189)^2 + (6-1)(152)^2}{4+6-2} = 27{,}835.$$

Hence, the estimated standard deviation of $D$ equals:

$$s_D = \sqrt{27{,}835\,(1/4 + 1/6)} = \sqrt{11{,}598} = 107.7 \text{ hours.}$$

The $t$ value for a 90% confidence interval and $4 + 6 - 2 = 8$ degrees of freedom is 1.860. Substituting the above values into the formula for the confidence interval for $\mu_2 - \mu_1$ gives:

$$(1243 - 985) - 1.860(107.7) \leqslant \mu_2 - \mu_1 \leqslant (1243 - 985) + 1.860(107.7),$$
$$\text{or: } 58 \leqslant \mu_2 - \mu_1 \leqslant 458.$$

Thus, we estimate that the mean service life of Type B bulbs exceeds that of Type A bulbs by between 58 and 458 hours (90% confidence coefficient).

**Problem 4**

A supermarket wishes to compare the mean time that customers spend in its store with the mean time spent by customers in its competitor's store. A simple random sample of twelve customers at each store is selected. Suppose the sample results are:

| Own Store | | Competitor's Store | |
|---|---|---|---|
| $\bar{X}_1 = 29$ minutes | $s_1 = 20$ minutes | $\bar{X}_2 = 35$ minutes | $s_2 = 12$ minutes |

Assuming that the times spent in each store are normally distributed with equal variances, construct a 98% confidence interval for the difference in mean times for the two stores. ✓

## REVIEW PROBLEMS

1. A contractor undertook four well-drilling operations in a region. The mean and standard deviation of the costs of these operations were \$8000 and \$2000 respectively. Assume that the cost of well drilling in the region is a normally distributed variable and that the four drilling operations represent a simple random sample from this normal distribution.
   (a) Construct a 90% confidence interval for the mean cost of a well-drilling operation in the region.
   _____ √
   (b) Refer to the confidence interval in (a). Does the confidence interval tell us that the cost of the next well-drilling operation in the region will fall within the confidence limits? _____
   _____ √

2. It is desired to construct a 99% confidence interval for the difference in the mean times required to perform a laboratory test using two different techniques. For each technique, a simple random sample of eight tests is timed. The sample results are:

| Technique I | | Technique II | |
|---|---|---|---|
| $\bar{X}_1 = 48$ minutes | $s_1 = 16$ minutes | $\bar{X}_2 = 52$ minutes | $s_2 = 12$ minutes |

   Assume that the times are normally distributed with equal standard deviations.
   (a) Construct the desired confidence interval. _____
   (b) Does the assumption of equal population standard deviations seem reasonable in this problem?
   _____
   _____ √

Answers

4. Pooled estimate $s^2 = \dfrac{11(20)^2 + 11(12)^2}{12 + 12 - 2} = 272$;

$s_D = \sqrt{272(1/12 + 1/12)} = 6.73$;
$t = 2.508$ for 22 degrees of freedom and a total tail area of .02.
$(35 - 29) - 2.508(6.73) \leqslant \mu_2 - \mu_1 \leqslant (35 - 29) + 2.508(6.73)$,
which gives $-10.88 \leqslant \mu_2 - \mu_1 \leqslant +22.88$.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $s_{\bar{X}} = 2000/\sqrt{4} = 1000$;
$t = 2.353$ for $4 - 1 = 3$ degrees of freedom and a total tail area of .10.
$8000 - 2.353(1000) \leqslant \mu \leqslant 8000 + 2.353(1000)$ or $5647 \leqslant \mu \leqslant 10{,}353$.

(b) The confidence interval in (a) applies to the *mean* cost of a drilling operation and not to the cost of any particular operation.

2. (a) Pooled estimate $s^2 = \dfrac{(8 - 1)(16)^2 + (8 - 1)(12)^2}{8 + 8 - 2} = 200$;

$s_D = \sqrt{200(1/8 + 1/8)} = 7.07$;
$t = 2.977$ for $8 + 8 - 2 = 14$ degrees of freedom and a total tail area of .01.
$(52 - 48) - 2.977(7.07) \leqslant \mu_2 - \mu_1 \leqslant (52 - 48) + 2.977(7.07)$,
which gives $-17.0 \leqslant \mu_2 - \mu_1 \leqslant 25.0$.

(b) The sample standard deviations, $s_1 = 16$ and $s_2 = 12$ in this case, provide us with evidence about whether or not $\sigma_1 = \sigma_2$. A statistical test discussed in Section 25 may be used to test the assumption of equal standard deviations. Without going into the details, it may be concluded that the sample standard deviations, 16 and 12, are sufficiently close to being equal that we may act on the assumption that $\sigma_1 = \sigma_2$.

# 14

# Interval Estimation of the Population Proportion – Large Simple Random Sample

In this section, we discuss the construction of a confidence interval for the population proportion, using a large simple random sample.

**A. Confidence Interval for Population Proportion.** The confidence interval for the population proportion $p$, when the sample size is large, is:

$$\bar{p} - z\sigma_{\bar{p}} \leqslant p \leqslant \bar{p} + z\sigma_{\bar{p}}.$$

Here $\bar{p}$ is the sample proportion, $z$ is the standard normal deviate corresponding to the specified confidence coefficient, and $\sigma_{\bar{p}}$ is the standard deviation of $\bar{p}$, that is, the standard deviation of the sampling distribution of $\bar{p}$. Recall that $\sigma_{\bar{p}} = \sqrt{p(1-p)/n}$ for infinite populations and $\sigma_{\bar{p}} = \sqrt{(N-n)/(N-1)} \cdot \sqrt{p(1-p)/n}$ for finite populations.

Since the value of $p$ is unknown, we use the sample proportion $\bar{p}$ in place of $p$ in the formula for $\sigma_{\bar{p}}$. Letting $s_{\bar{p}}$ denote the estimator of $\sigma_{\bar{p}}$, we have $s_{\bar{p}} = \sqrt{\bar{p}(1-\bar{p})/(n-1)}$ for infinite populations and $s_{\bar{p}} = \sqrt{(N-n)/(N-1)} \cdot \sqrt{\bar{p}(1-\bar{p})/(n-1)}$ for finite populations. Note that the denominator now is $n-1$ rather than $n$. (In some books this change is not made. In the large sample case, the effect of using $n$, rather than $n-1$, is small.) Substituting $s_{\bar{p}}$ for $\sigma_{\bar{p}}$ in the above confidence interval gives:

$$\bar{p} - zs_{\bar{p}} \leqslant p \leqslant \bar{p} + zs_{\bar{p}}.$$

When the sample size is large, as is assumed here, the use of $s_{\bar{p}}$ instead of $\sigma_{\bar{p}}$ has no appreciable effect on the confidence coefficient.

Example 1 An insurance company wishes to construct a 95.4% confidence interval for the proportion of policyholders who have more than one policy with the company. Assume that the number of policyholders is very large. In a simple random sample of 170 policyholders, it is found that 26% have more than one policy. For this problem, $n = 170$, $\bar{p} = .26$, $s_{\bar{p}} = \sqrt{(.26)(.74)/(170-1)} = .034$, and $z = 2$. We obtain for the confidence interval:

$$.26 - 2(.034) \leqslant p \leqslant .26 + 2(.034), \quad \text{which gives } .192 \leqslant p \leqslant .328.$$

Hence, we estimate that between 19.2% and 32.8% of the company's policyholders have more than one policy with the company. This estimate is made with a confidence coefficient of 95.4%.

**Problem 1**

In a meat packing plant, an inspection of a simple random sample of 50 pork bellies indicates that 10 have excess amounts of leaf lard.

(a) Construct a 90% confidence interval for the proportion of bellies in the population with excess leaf lard. (Assume the population is very large.) _____ √

(b) In the past, 8% of pork bellies had excess leaf lard. Would you conclude from the interval estimate in (a) that this proportion has changed? _____

_____ √

**B. Determination Of Necessary Sample Size.** The planning of the sample size for estimating a population proportion corresponds to that for estimating a population mean. If it is desired to estimate the population proportion within $\pm h$, with a specified confidence coefficient (expressed in terms of $z$), the necessary sample size is given by:

$$n = \frac{z^2 p(1-p)}{h^2} \text{ for infinite populations,}$$

$$n = \frac{p(1-p)}{\frac{h^2}{z^2} + \frac{p(1-p)}{N}} \text{ for finite populations.}$$

Before we can solve for $n$, it is necessary to specify a planning value of $p$ to be used in the sample size formula.

Example 2 The Acme Company wishes to estimate the proportion of time a computer facility shared with another company is used by Acme. A random sample of operating times is to be selected throughout the year, and the proportion of the times that Acme is using the facility is to be recorded. It is desired to have an estimate of this proportion within ±.02, with a confidence coefficient of 95.4%. In the past, Acme's utilization of the facility has amounted to approximately 60% of the operating time.

To obtain the necessary sample size, the sample size formula for an infinite population (very large population) is used with $z = 2$, $p = .60$, and $h = .02$. Substituting into the formula gives $n = (2)^2(.60)(.40)/(.02)^2 = 2400$. Hence, 2400 random operating times throughout the year need to be observed.

**Problem 2**

A research group wishes to determine the proportion of dentists who are in favor of proposed legislation which affects dental practice. The dentists in the country are to be sampled and an estimate is to be obtained of the true proportion within ±.03, with a confidence coefficient of 99.7%. The research group is certain that the proportion is less than .60 but is not certain how much less than .60 it is. Assume that the population is large relative to the sample sizes.

(a) Determine the sample size if the planning value for $p$ is

(i) .3: _____

(ii) .5: _____

(iii) .6: _____ √

(b) What generalization does this suggest? _____

_____

(c) What planning value for $p$ should be used if the sample size should rather be too large than too small?

_____

(d) Indicate the sample size which will guarantee the specified precision. _____ √

(e) Suppose the research group uses the sample size suggested in (d) and obtains the result, $\bar{p} = .36$. Calculate the confidence interval for the population proportion. _____

_____

(f) Does the confidence interval in (e) have the desired precision? _____

_____

(g) What planning value of $p$ should be used if the research group is certain that the population proportion is at least .6? _____

_____ √

**C. Confidence Interval for Difference Between Two Population Proportions.** Frequently, one desires a confidence interval for the difference between two population proportions. If large, independent simple random samples from the respective populations under study are used, the confidence interval has the form:

$$d - z\sigma_d \leqslant p_2 - p_1 \leqslant d + z\sigma_d.$$

Here $d = \bar{p}_2 - \bar{p}_1$, where $\bar{p}_2$ and $\bar{p}_1$ are the sample proportions for the two samples, and $\sigma_d = \sqrt{\sigma^2_{\bar{p}_1} + \sigma^2_{\bar{p}_2}}$, where $\sigma_{\bar{p}_1}$ and $\sigma_{\bar{p}_2}$ are the standard deviations of $\bar{p}_1$ and $\bar{p}_2$, based on samples of size $n_1$ and $n_2$ respectively.

The values of $p_1$ and $p_2$ are not known, hence $\sigma_{\bar{p}_1}$ and $\sigma_{\bar{p}_2}$ are unknown also. However, since $n_1$ and $n_2$ are large, no significant effect on the confidence coefficient is introduced by replacing $\sigma_d$ with $s_d$, where $s_d = \sqrt{s^2_{\bar{p}_1} + s^2_{\bar{p}_2}}$. Recall that $s_{\bar{p}} = \sqrt{\bar{p}(1 - \bar{p})/(n \quad 1)}$ for infinite or very large populations. Substituting $s_d$ for $\sigma_d$, the confidence interval becomes:

$$d - zs_d \leqslant p_2 - p_1 \leqslant d + zs_d.$$

Example 3 A simple random sample of 730 items from a large shipment from Supplier A contains 50 unsatisfactory items. An independent simple random sample of 785 items from a large shipment from Supplier B contains 145 unsatisfactory items. It is desired to construct a 90% confidence interval for the difference between the proportions of items in the two shipments which are unsatisfactory. The construction of the confidence interval proceeds as follows:

| Supplier A | | Supplier B | |
|---|---|---|---|
| $\bar{p}_1$ | $= 50/730 = .068$ | $\bar{p}_2$ | $= 145/785 = .185$ |
| $s_{\bar{p}_1}$ | $= \sqrt{(.068)(.932)/(730 - 1)} = .0093$ | $s_{\bar{p}_2}$ | $= \sqrt{(.185)(.815)/(785 - 1)} = .0139$ |
| $n_1$ | $= 730$ | $n_2$ | $= 785$ |

$d = \bar{p}_2 - \bar{p}_1 = .185 - .068 = .117$, $s_d = \sqrt{(.0093)^2 + (.0139)^2} = .0167$, and $z = 1.64$ corresponds to a 90% confidence coefficient:

$$.117 - (1.64)(.0167) \leqslant p_2 - p_1 \leqslant .117 + (1.64)(.0167)$$

which gives $.090 \leqslant p_2 - p_1 \leqslant .144$.

Thus, the per cent of unsatisfactory items in the shipment from Supplier B exceeds that from Supplier A by between 9.0 and 14.4 per cent points (90% confidence coefficient).

**Problem 3**

Simple random samples, each of 100 units of product selected from the day shift's and night shift's production, contained respectively 20 and 10 units with defective packaging.

(a) Construct a 95% confidence interval for the difference in the proportion of units with defective packaging for the day ($p_2$) and night ($p_1$) shifts. _____

_____

(b) Does your confidence interval provide any indication of whether or not the night shift's packaging operation differs from that of the day shift with respect to the proportion of units with defective packaging?

_____ √

Answers

1. (a) $\bar{p} = 10/50 = .20$, $s_{\bar{p}} = \sqrt{(.20)(.80)/(50-1)} = .057$, and $z = 1.64$; $.20 - 1.64(.057) \leqslant p \leqslant .20 + 1.64(.057)$, which gives $.107 \leqslant p \leqslant .293$.

   (b) Since the confidence interval in (a) does not include $p = .08$, at the 90% confidence level, one would conclude that the proportion has changed.

2. (a) (i) $n = \dfrac{3^2(.3)(.7)}{(.03)^2} = 2100$;

   (ii) $n = \dfrac{3^2(.5)(.5)}{(.03)^2} = 2500$;

   (iii) $n = \dfrac{3^2(.6)(.4)}{(.03)^2} = 2400$.

   (b) As the planning value of $p$ approaches .5 from either direction, the necessary sample size takes on its largest value, 2500 in this case.

   (c) $p = .5$.

   (d) **2500** [see (a) (ii)].

   (e) $\bar{p} = .36$, $s_{\bar{p}} = \sqrt{(.36)(.64)/(2500-1)} = .0096$, and $z = 3$;

   $.36 - 3(.0096) \leqslant p \leqslant .36 + 3(.0096)$, which gives $.3312 \leqslant p \leqslant .3888$.

   (f) **Yes, the confidence interval meets the desired precision requirement; the half-width is .0288 compared with the specified half-width, .03.**

   (g) **The planning value should be .6 since this is the closest value to .5 among the eligible values.**

3. (a)

| Night Shift | Day Shift |
|---|---|
| $\bar{p}_1 = 10/100 = .1$ | $\bar{p}_2 = 20/100 = .2$ |
| $s_{\bar{p}_1} = \sqrt{(.1)(.9)/(100-1)}$ | $s_{\bar{p}_2} = \sqrt{(.2)(.8)/(100-1)}$ |

   $d = .2 - .1 = .1$, $s_d = \sqrt{(.1)(.9)/(100-1) + (.2)(.8)/(100-1)} = .050$, and $z = 1.96$;

   $.1 - 1.96(.050) \leqslant p_2 - p_1 \leqslant .1 + 1.96(.050)$, which gives $.002 \leqslant p_2 - p_1 \leqslant .198$.

   (b) **Yes, the proportion of units with defective packaging is between .2 and 19.8 per cent points greater for the day shift than the night shift (95% confidence coefficient).**

## REVIEW PROBLEMS

1. Refer to Example 2. Of the 2400 random operating times when the computer facility was observed, Acme was found to be using it 1536 times.
   (a) Construct a 95.4% confidence interval for the proportion of time that Acme used the computer facility during the year. ✓
   (b) Does the estimate of the proportion have the desired precision ±.02 at the 95.4% confidence level?
   (c) If the computer facility costs $310,000 for the year, and the cost is to be divided between the two companies according to the per cent of time used by each, how much of this cost should be charged to Acme?
   (d) Based on your confidence interval, what is the smallest amount that Acme might have been charged if a complete record of facility utilization had been maintained?
   (e) The two firms agree to use the work-sampling procedure again next year. They still desire to have an estimate of the true proportion within ±.02 with a confidence coefficient of 95.4%. However, a change in Acme's operations makes it uncertain what proportion of time Acme will require the computer facility in the coming year. Suggest a sample size for the number of observations which should be made on the facility under these new circumstances to guarantee the desired precision. ✓

2. Two cash register clerks were studied independently to compare the proportions of items each registers incorrectly. In a simple random sample of 145 items, it was found that Clerk A registered five items incorrectly. In a simple random sample of 170 items, it was found that Clerk B registered four items incorrectly. Construct a 99.7% confidence interval for the difference in the proportion of items $(p_A - p_B)$ registered incorrectly by the two clerks. ✓

## ANSWERS TO REVIEW PROBLEMS

1. (a) $\bar{p} = 1536/2400 = .64$, $s_{\bar{p}} = \sqrt{(.64)\,(.36)/(2400-1)} = .0098$, and $z = 2$;
$.64 - 2(.0098) \leqslant p \leqslant .64 + 2(.0098)$, which gives $.62 \leqslant p \leqslant .66$.

   (b) **Yes, the half-width of the interval in (a) is .0196.**

   (c) Using the point estimate $\bar{p} = .64$, $\$310{,}000 \times .64 = \$198{,}400$.

   (d) **The lower limit in the confidence interval in (a) is .62. Hence, $\$310{,}000 \times .62 = \$192{,}200$.**

   (e) Use an advance estimate of $p = .5$ in order to assure that the sample size is large enough regardless of the actual value of $p$.
   Hence, $p = .5$, $z = 2$, $h = .02$, gives:

$$n = (2)^2\,(.5)(.5)/(.02)^2 = 2500.$$

   A sample size of 2500 should be used in the coming year.

2.

| Clerk B | Clerk A |
|---|---|
| $\bar{p}_B = 4/170 = .0235$ | $\bar{p}_A = 5/145 = .0345$ |
| $s_{\bar{p}_B} = \sqrt{(.0235)(.9765)/(170-1)} = .0117$ | $s_{\bar{p}_A} = \sqrt{(.0345)(.9655)/(145-1)} = .0152$ |

$d = .0345 - .0235 = .011$, $s_d = \sqrt{(.0117)^2 + (.0152)^2} = .019$, and $z = 3$;

$.011 - 3(.019) \leqslant p_A - p_B \leqslant .011 + 3(.019)$, which gives $-.046 \leqslant p_A - p_B \leqslant .068$.

# 15

# Introduction to Decision-Making

Decision-making plays a central role in many areas. In this section we will consider the basic form of the decision problem.

**A. Decision-Making Under Uncertainty.** Decision problems have some basic common characteristics. In any decision problem, there are two or more courses of action or *acts* available to the decision-maker. Each act entails some results or *consequences*, which in general differ from act to act. The task of the decision-maker is to select the act that is best in the light of the *criterion*, or basis of choice, which he applies to the consequences of the acts.

In decision-making under uncertainty, the consequences of each act depend on a variable (or several variables) outside the control of the decision-maker. The possible values this variable may take on are called the *states of nature* or *outcome states* of the decision problem. In decision-making under uncertainty, the decision-maker has no knowledge about the outcome state that will prevail until after he has selected and implemented an act.

**Example 1** Every Monday the Standard Co. must decide how many units it will produce that week in order to fill the order of a special customer. The actual order is not received until Wednesday. The order may be for 0, 1, or 2 units. If too many units are produced, the excess is sold below cost locally since storage is expensive. If too few are produced, a production run must be set up at overtime cost to complete the order.

This decision problem is described in Table 15.1, called a *payoff table*.

**Table 15.1**

| | | Units Ordered | | |
|---|---|---|---|---|
| | | 0 | 1 | 2 |
| Units Produced | 0 | \$0 | 60 | 120 |
| | 1 | −10 | 100 | 160 |
| | 2 | −20 | 90 | 200 |

The three production levels (rows) represent the different acts available to the Standard Co., and the three order amounts (columns) represent the different possible outcome states. The values in the cells represent the consequences or *payoffs* (profits in this case) that result from the selection of a particular act and the realization of a given outcome state. For example, if one unit is produced and the customer orders two, then a profit of \$160 is realized.

**Problem 1**

A firm intends to manufacture a style item. The profit depends on the number of years the style item remains popular, which can be 0, 1, 2, or 3 years. Four types of production facilities may be used, denoted by $A$, $B$, $C$, and $D$. The firm must select one of the four types of facilities to produce the item.

For this problem, describe:

(a) acts: _____

_____

(b) outcome states: _____

_____

(c) payoffs: _____

_____

_____ √

**B. Admissible Acts.** An admissible act is an act that is not dominated by another act. One act dominates another if for each possible outcome state the first act yields a payoff at least as good as that of the second and yields a higher payoff under at least one outcome state. Ordinarily, only admissible acts need be considered in a decision problem.

Example 2 In Example 1, all three acts are admissible. For instance, the act "produce 0" has higher profits than the act "produce 1" if 0 units are ordered, but the reverse holds if 1 unit is ordered. Thus, neither of these two acts dominates the other. Similarly, the other pairs of acts can be analyzed. Since no act dominates any other, all are admissible.

**Problem 2**

Refer to Problem 1. Assume that the following payoff table is appropriate for the decision problem. (The payoffs represent the present value of estimated profits in millions of dollars.)

**Table 15.2**

| | | Years of Popularity (outcome states) | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 |
| Type of | *A* | −1 | 2 | 2 | 2 |
| Production | *B* | −3 | 0 | 3 | 5 |
| Facility | *C* | −5 | −2 | 2 | 8 |
| (acts) | *D* | −5 | 0 | 2 | 5 |

Which of the acts in this decision problem are admissible? _____

_____ √

**C. Possible Criteria for Selecting an Act.**

Example 3 To illustrate a number of criteria that have been proposed for decision-making under uncertainty and to show the application of the criteria, we use the situation described in Example 1. Note how the determination as to which act is "best" depends on the criterion employed.

*Maximax criterion.* Consider the maximum payoff possible with each act. Under the maximax criterion, the best act is the one that maximizes the maximum payoff. In Example 1, it can be seen that the highest profit when producing two units is $200 (if an order for two units is received). Neither of the other acts has a maximum payoff this high for any outcome state. Hence, the act "produce 2" is optimal according to the maximax criterion.

*Maximin criterion.* Consider the minimum payoff possible with each act. Under the maximin criterion, the best act is the one which maximizes the minimum payoff. In Example 1, the minimum payoffs corresponding to the acts, produce zero, one, and two units are $0, −$10, and −$20, respectively. Producing zero units therefore maximizes the minimum possible payoff.

*Minimax regret criterion.* This criterion focuses attention on opportunity losses. The opportunity loss for a given act is the difference between the best payoff that could be obtained under a given outcome state and the actual payoff of that act under this state. Consider Example 1. If an order for two units is ultimately received, the

best payoff would have been obtained by producing two units for a profit of \$200. If, however, one unit had been produced, the payoff would be only \$160. Hence the act of producing one unit when an order for two is ultimately received results in an opportunity loss of 200 − 160 = \$40. Similarly, the opportunity loss may be obtained for every other cell of the payoff table. The opportunity losses for Example 1 are shown in Table 15.3. Frequently, the opportunity losses are called *regrets*, and then Table 15.3 is referred to as a *regret table.*

**Table 15.3**

| | | Units Ordered | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | Maximum Regret Value |
| Units Produced | 0 | 0 | 40 | 80 | 80 |
| | 1 | ___ | 0 | 40 | ___ |
| | 2 | 20 | ___ | 0 | ___ |

The minimax regret criterion considers the maximum regret possible with each act. For instance, for the act "produce 0", the maximum regret value is 80. The maximum regrets are shown to the right of the regret table. The act that minimizes the maximum regret is considered the best according to the minimax regret criterion.

**Problem 3**

(a) Fill in the blank regret values in Table 15.3.
(b) Fill in the blank maximum regret values.
(c) Which act has the smallest maximum regret value? _____
(d) Which act is optimal according to the minimax regret criterion? _____ √

**Problem 4**

For Table 15.2:
(a) What are the maximum profits for the different acts? _____
(b) Which is the best act according to the maximax criterion? ____
(c) What are the minimum profits for the different acts? _____
(d) Which is the best act according to the maximin criterion? ____
(e) Develop the regret table.

(f) Which is the best act according to the minimax regret criterion? ____ √

**D. Decision-Making Under Risk (Expected Payoff Criterion).** In problems involving decision-making under risk, the decision-maker is able to assign probabilities to each of the outcome states even though he is uncertain as to which state will actually occur. In these instances, the expected payoff criterion frequently is used to select an act.

Example 4 Consider Example 1. Suppose the Standard Co. has reliable information from past experience which indicates that the probabilities of orders for zero, one, and two units are .5, .4, and .1 respectively. The payoff table and these probabilities are shown in Table 15.4.

Answers

1. (a) **Acts: the selection of any one of the four types of production facilities.**
   (b) Outcome states: the possible number of years that the item remains popular.
   (c) Payoffs: the profits that result from using a given type of production facility in combination with the number of years that the item remains popular.

2. All of the acts are admissible except *D*, which is dominated by act *B*.

3. (a) (b)

| | | Units Ordered | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | Maximum Regret Value |
| Units Produced | 0 | 0 | 40 | 80 | 80 |
| | 1 | 10 | 0 | 40 | 40 |
| | 2 | 20 | 10 | 0 | 20 |

   (c) **"Produce 2."**
   (d) **The act with the smallest maximum regret value, "produce 2" in this case.**

4. (a) *A*: 2; *B*: 5; *C*: 8; *D*: 5.
   (b) *C*.
   (c) *A*: −1; *B*: −3; *C*: −5; *D*: −5.
   (d) *A*.
   (e)

| | | Years of Popularity | | | | |
|---|---|---|---|---|---|---|
| | | 0 | 1 | 2 | 3 | Maximum Regret Value |
| Type of Production Facility | *A* | 0 | 0 | 1 | 6 | 6 |
| | *B* | 2 | 2 | 0 | 3 | 3 |
| | *C* | 4 | 4 | 1 | 0 | 4 |
| | *D* | 4 | 2 | 1 | 3 | 4 |

   (f) *B*.
   Since act *D* is dominated, it would not be selected under any one of the three criteria. Thus it may be ignored completely.

**Table 15.4**

| | | Units Ordered | | | |
|---|---|---|---|---|---|
| | | 0 | 1 | 2 | |
| | Probability: | .5 | .4 | .1 | Expected Payoff |
| Units Produced | 0 | $0 | 60 | 120 | $36 |
| | 1 | −10 | 100 | 160 | 51 |
| | 2 | −20 | 90 | 200 | 46 |

The *expected payoff criterion* identifies as best the act that has the maximum expected payoff. For instance, for the act of producing zero items, the expected payoff is equal to $0(.5) + 60(.4) + 120(.1) = \$36$. The expected payoff for each of the acts is shown to the right of Table 15.4. The act of producing one unit maximizes the expected payoff, hence is best under the expected payoff criterion.

**Problem 5**

Consider Table 15.2. Assume the firm assigns the following probabilities to each number of years of popularity: zero years, .1; one year, .4; two years, .3; three years, .2. Given these probabilities:

(a) Obtain the expected payoff for each admissible act. $A$: _____ ; $B$: _____ ; $C$: _____ .

(b) Which act is optimal according to the expected payoff criterion? ___√

**E. Perfect Information.** In decision-making under risk, a decision-maker may have access to information about the outcome states that reduces the risk inherent in his decision problem. Perfect information about the outcome state identifies the actual outcome state that will prevail. Here we consider the value of perfect information to the decision-maker.

Example 5 Consider Example 1. If the decision-maker has access to perfect information in this problem, then if zero units will be ordered, the optimal act is to produce zero units; if one unit will be ordered, to produce one unit; if two units will be ordered, to produce two units. The payoffs corresponding to these three optimal acts are \$0, \$100, and \$200 respectively. The probabilities that the order will be for zero, one, or two units are .5, .4, and .1 respectively. Hence, the expected payoff with perfect information, assuming it is free, equals: 0(.5) + 100(.4) + 200 (.1) = \$60. In Example 4, it was shown that with only the probability information about the outcome states, the expected payoff of the optimal act is \$51. Hence, the expected value of perfect information is 60 – 51 = \$9.

**Problem 6**

Consider Example 4. Since the customer's order is sent by mail every Monday morning, arriving at the Standard Co. on Wednesday, an executive of the Standard Co. reasons that a phone call to the customer every Monday morning would provide perfect information about the number of units being ordered by the customer.

(a) If the phone call would cost \$3, should the Standard Co. proceed to phone for the order every Monday morning? _____

(b) What is the expected payoff per week if the phone call is made each week? _____√

(c) What if the phone call costs \$15? _____

(d) Suppose a market researcher offers to sell reliable but not perfect information for \$10 per week as to the number of units to be ordered, and assume no perfect information is available at any price. Should the market research service be purchased? _____

(e) If the probabilities in Example 4 were different, might the expected payoff with perfect information be different? _____

Might the expected value of perfect information be different? _____ √

**Problem 7**

Refer to Problem 2. Using the probabilities given in Problem 5, determine:

(a) The expected payoff with perfect, free, information. _____

(b) The expected value of perfect information. _____√

Answers

5. (a)

| Act | | Expected Payoff |
|---|---|---|
| *A* | $-1(.1) + 2(.4) + 2(.3) + 2(.2) =$ | 1.7 |
| *B* | | 1.6 |
| *C* | | .9 |

(b) *A*.

6. (a) Since the expected value of perfect information exceeds the cost of the phone call ($\$9 > \$3$), the Standard Co. should proceed to phone for the order every Monday morning.
(b) $60 - 3 = \$57$.
(c) Since the cost of the phone call exceeds the expected value of perfect information ($\$15 > \$9$), the Standard Co. should not phone for the order every Monday morning.
(d) No. Because the expected value of imperfect information cannot exceed the expected value of perfect information, the company should not pay more than $9 for any information.
(e) If the prior probabilities were different, the expected payoff with perfect information might be changed. Further, the optimal act and/or the expected payoff of the optimal act might be changed. Therefore, the expected value of perfect information might be changed either by the change in the expected payoff with perfect information or by the change in the expected payoff of the optimal act or both.

7. (a) $-1(.1) + 2(.4) + 3(.3) + 8(.2) = 3.2$.
(b) Expected payoff of the optimal act = 1.7. (See the answer to Problem 5(a).) Hence, the expected value of perfect information = $3.2 - 1.7 = 1.5$.

## REVIEW PROBLEMS

1. A firm must decide whether it will use no inspection, limited inspection, or complete (100%) inspection to determine the quality of an incoming shipment of raw material. The payoff to the firm depends on the quality of the incoming shipment. Let $Q_1$, $Q_2$, $Q_3$, and $Q_4$ describe the four possible quality levels of the shipment. Assume that the following payoff table (the payoffs represent gains or profits) is appropriate for this decision problem:

**Table 15.5**

| | | Quality | | | |
|---|---|---|---|---|---|
| | | $Q_1$ | $Q_2$ | $Q_3$ | $Q_4$ |
| Inspection Plan | None | 240 | 200 | 150 | 50 |
| | Limited | 230 | 195 | 150 | 60 |
| | Complete | 150 | 150 | 150 | 150 |

(a) For Table 15.5, indicate the acts that would be selected with each of the following criteria:
   (i) maximax: _____
   (ii) maximin: _____
   (iii) minimax regret: _____ √

(b) Based on the quality levels of previous shipments, the quality control inspector assesses the probabilities of the various quality levels to be $P(Q_1) = .1, P(Q_2) = .5, P(Q_3) = .3$, and $P(Q_4) = .1$.
   (i) Using these probabilities, which act is selected by the expected payoff criterion? _____
   _____
   (ii) What is the expected value of perfect information in this problem? _____
   (iii) Which act is optimal if the expected regret is to be minimized? _____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a) (i) No inspection.
   (ii) Complete inspection.
   (iii)

| | Regret Table | | | | Maximum Regret Value |
|---|---|---|---|---|---|
| None | 0 | 0 | 0 | 100 | 100 |
| Limited | 10 | 5 | 0 | 90 | <u>90</u> |
| Complete | 90 | 50 | 0 | 0 | <u>90</u> |

The decision-maker should choose either the limited or complete inspection plan.

(b) (i)

| Act | Expected Payoff |
|---|---|
| None | <u>174</u> |
| Limited | 171.5 |
| Complete | 150 |

Under the expected payoff criterion, "no inspection" is the optimal act.

(ii) Expected payoff with perfect, free, information = 240(.1) + 200(.5) + 150(.3) + 150(.1) = 184. Expected payoff of the optimal act = 174. (See (b)(i) above.) Hence, the expected value of perfect information = 184 − 174 = 10.

(iii) The regret table for this problem is shown in the answer to Review Problem 1(a)(iii). The expected regret values for the acts are:

| Act | | Expected Regret |
|---|---|---|
| None | 0(.1) + 0(.5) + 0(.3) + 100(.1) = | <u>10</u> |
| Limited | 10(.1) + 5(.5) + 0(.3) + 90(.1) = | 12.5 |
| Complete | 90(.1) + 50(.5) + 0(.3) + 0(.1) = | 34 |

The act "no inspection" minimizes the expected regret.

# 16

# Statistical Decision-Making: Hypothesis Testing – Population Mean, Large Sample

**A. The Decision Problem.** In *statistical* decision-making, *sample* information is used to select the appropriate act from the acts available to the decision-maker. The outcome states for the decision problem correspond to the possible values which may be taken on by the relevant parameter of the population under study.

In this section, we consider statistical decision problems with the following characteristics.

(i) Two alternative acts are available to the decision-maker. These acts are labelled $C_1$ and $C_2$.

(ii) The relevant parameter is the population mean $\mu$ and the outcome states correspond to the possible values of $\mu$.

(iii) The sample mean $\bar{X}$ is used to select the appropriate act.

(iv) The sample mean is obtained from a large, simple random sample of size $n$.

This type of decision problem often is called *hypothesis testing*. In this terminology, the two alternatives $C_1$ and $C_2$ correspond to two alternative hypotheses.

Example 1 A firm periodically receives large shipments of rough shafts that require machining to fine tolerances. The payoff to the firm is affected by the mean weight of the rough shafts in the shipment, since this largely determines the amount of metal to be removed. The firm has found from a study of costs that if the mean weight of the shafts in the shipment does not exceed 26 pounds, the machining should be done in the house. If the mean weight does exceed 26 pounds, the machining should be subcontracted out.

For this problem, the population mean $\mu$ is the mean weight of the shafts in the shipment, and the acts $C_1$ and $C_2$ may be defined in two equivalent ways as follows:

$C_1$: do the machining in the house, $C_1$: $\mu \leqslant 26$ pounds,

OR

$C_2$: subcontract the machining out, $C_2$: $\mu > 26$ pounds.

The reason for the equivalence is that the value of $\mu$ immediately determines which act is desired. We shall use $C_1$ and $C_2$ here either to denote the respective acts "do the machining in the house" and "subcontract the machining out" or to denote conclusions or hypotheses about the population characteristic, namely, $\mu \leqslant 26$ and $\mu > 26$ respectively.

(In some books, the symbols $H_1$ and $H_2$, or $H_0$ and $H_1$, are used instead of $C_1$ and $C_2$ to denote the alternative acts, conclusions, or hypotheses.)

**Problem 1**

An electronic tube manufacturer claims that his product has a mean service life greater than 200 hours. A firm that requires a large number of these tubes wishes to test this claim. If the claim is correct, they will buy his product; otherwise they will go to another supplier.

For this decision problem, define $C_1$ and $C_2$ in the two equivalent forms.

$C_1$: seek another supplier OR $C_1$: _____

$C_2$: _____ $C_2$: _____ √

**Problem 2**

A purchasing agent for a cafeteria chain purchases broiling chickens in very large shipments. He is willing to accept shipments in which the mean weight of the broilers is greater than 32 ounces but will seek a price reduction from the vendor if the mean weight of the broilers is less than or equal to 32 ounces.

For this decision problem, define $C_1$ and $C_2$ in the two equivalent forms.

$C_1$: seek a price reduction OR $C_1$: _____

$C_2$: _____ $C_2$: _____ √

**B. Types of Errors.** Two types of errors can be made in any statistical decision problem involving two acts or conclusions. A *Type I error* is to conclude $C_2$ when $C_1$ is correct, and a *Type II error* is to conclude $C_1$ when $C_2$ is correct. In some books, a Type I error is referred to as an $\alpha$ error and a Type II error is referred to as a $\beta$ error.

Example 2 For the illustration in Example 1, a Type I error is made if the machining is subcontracted out when it should have been done in the house, i.e., if it is concluded that $\mu > 26$ when actually $\mu \leqslant 26$. A Type II error is made if the machining is done in the house when it should have been subcontracted out, i.e., if it is concluded that $\mu \leqslant 26$ when actually $\mu > 26$. These two types of errors are illustrated in the following table:

| | Outcome State | |
|---|---|---|
| Conclusion Reached | $\mu \leqslant 26$ | $\mu > 26$ |
| $C_1$: $\mu \leqslant 26$ (in house) | Correct conclusion | Type II error |
| $C_2$: $\mu > 26$ (subcontract) | Type I error | Correct conclusion |

**Problem 3**

(a) For Problem 1, construct a table of the type shown in Example 2 above. √

(b) In words, define a Type I error and a Type II error in Problem 1.

Type I error: _____

Type II error: _____ √

**Problem 4**

(a) For Problem 2, construct a table of the type shown in Example 2 above. √

(b) In words, define a Type I error and a Type II error in Problem 2.
Type I error: _____
Type II error: _____ √

**C. Statistical Decision Rule or Statistical Test.** A statistical decision rule specifies, for each possible outcome in the sample space, the conclusion to be reached. For the problems considered here, the sample space consists of the possible values of the sample mean $\overline{X}$. (In some books, the decision rule involves $(\overline{X} - \mu)/\sigma_{\overline{X}}$. This modification does not affect the properties of the decision rule to be discussed here.)

Example 3 Consider the illustration in Example 1. To obtain information about the mean weight of the shafts in the shipment, a simple random sample of 49 shafts is selected and the mean weight $\overline{X}$ is determined. The following is a statistical decision rule:

If $\overline{X} \leqslant 26.3$, conclude $C_1$ $(\mu \leqslant 26)$, that is, do the machining in the house.
If $\overline{X} > 26.3$, conclude $C_2$ $(\mu > 26)$, that is, subcontract the machining out.

The value 26.3 is called the *action limit* of the decision rule. Note how the decision rule indicates, for every possible value of the sample mean $\overline{X}$, whether conclusion $C_1$ or conclusion $C_2$ should be drawn.

**Problem 5**

(a) For the illustration in Example 1, indicate, for each of the following, whether or not it is a statistical decision rule.
(i) If $\overline{X} \leqslant 26.0$, conclude $C_1$; if $\overline{X} > 26.0$, conclude $C_2$. _____
(ii) If $\overline{X} \leqslant 26.0$, conclude $C_1$; if $\overline{X} > 26.3$, conclude $C_2$. _____
(iii) If $\overline{X} < 26.5$, conclude $C_1$; if $\overline{X} \geqslant 26.0$, conclude $C_2$. _____
(iv) Regardless of the value of $\overline{X}$, always conclude $C_1$, that is, always do the machining in the house.
_____ √

(b) For the decision rule in (a) (i), what conclusion should be drawn if $\overline{X} = 26.2$? ____ √

**D. Conclusion Probabilities or Act Probabilities.** The probabilities that a statistical decision rule will lead to each of the two conclusions, $C_1$ and $C_2$, for a given outcome state $\mu$, are called conclusion probabilities or act probabilities. These probabilities are obtained from the sampling distribution of $\overline{X}$ corresponding to particular values of $\mu$.

The probability of concluding $C_1$, given a specified value of $\mu$, will be denoted $P(C_1 : \mu)$. Similarly, the probability of concluding $C_2$, given a specified value of $\mu$, will be denoted $P(C_2 : \mu)$.

Example 4 Consider the decision rule in Example 3. Suppose that the mean weight of the shafts in the shipment is $\mu = 25.9$ pounds. Further assume that the standard deviation of the weights of the shafts is known from past experience to be $\sigma = 1.4$ pounds. From the decision rule it is seen that:

$$P(C_1 : \mu = 25.9) = P(\overline{X} \leqslant 26.3 : \mu = 25.9);$$

$$P(C_2 : \mu = 25.9) = P(\overline{X} > 26.3 : \mu = 25.9).$$

To obtain these two conclusion probabilities, it is necessary to consider the sampling distribution of $\overline{X}$. Since $\mu = 25.9$, $\sigma = 1.4$, and $n = 49$, it follows that the sampling distribution of $\overline{X}$ is approximately normal with a mean equal to 25.9 and a standard deviation equal to $\sigma/\sqrt{n} = 1.4/\sqrt{49} = .2$. The $z$ value corresponding to $\overline{X} = 26.3$ is $z = (26.3 - 25.9)/.2 = 2$. Thus, from the standard normal probability table, Table A-1, it follows that:

$$P(C_1 : \mu = 25.9) = P(\overline{X} \leqslant 26.3 : \mu = 25.9) = P(z \leqslant 2) = .9772;$$

$$P(C_2 : \mu = 25.9) = P(\overline{X} > 26.3 : \mu = 25.9) = P(z > 2) = .0228.$$

Answers

1. $\mu$: the mean service life of the manufacturer's tubes.

   $C_1$: seek another supplier, $C_1$: $\mu \leqslant 200$,

   OR

   $C_2$: buy his product, $C_2$: $\mu > 200$.

**If your answer is correct you may omit Problem 2 and go to Topic B: Otherwise do Problem 2.**

2. $\mu$: the mean weight of the broilers in the shipment.

   $C_1$: seek a price reduction, $C_1$: $\mu \leqslant 32$,

   OR

   $C_2$: accept the shipment, $C_2$: $\mu > 32$.

3. (a)

| | Outcome State | |
|---|---|---|
| Conclusion Reached | $\mu \leqslant 200$ | $\mu > 200$ |
| $C_1$: $\mu \leqslant 200$ | Correct conclusion | Type II error |
| $C_2$: $\mu > 200$ | Type I error | Correct conclusion |

(b) **Type I error: To buy the manufacturer's product when another supplier should have been sought.**
**Type II error: To seek another supplier when the manufacturer's product should have been purchased.**

**If your answer is correct you may omit Problem 4 and go on to Topic C: Otherwise do Problem 4.**

4. (a)

| | Outcome State | |
|---|---|---|
| Conclusion Reached | $\mu \leqslant 32$ | $\mu > 32$ |
| $C_1$: $\mu \leqslant 32$ | Correct conclusion | Type II error |
| $C_2$: $\mu > 32$ | Type I error | Correct conclusion |

(b) **Type I error: To accept the shipment when a price reduction should have been sought.**
**Type II error: To seek a price reduction when the shipment should have been accepted.**

5. (a) (i) **This is a statistical decision rule.**
   (ii) This is not a statistical decision rule. The rule does not specify the conclusion to be reached when $26.0 < \overline{X} \leqslant 26.3$.
   (iii) This is not a statistical decision rule. The rule is ambiguous because it states that both conclusions should be drawn when $26.0 \leqslant \overline{X} < 26.5$.
   (iv) **This is a statistical decision rule. In effect, this rule ignores the sample information.**

   (b) Since $26.2 > 26.0$, conclusion $C_2$ should be drawn.

Figure 16.1(a) shows the conclusion probabilities for the decision rule given in Example 3 when $\mu = 25.9$ pounds.

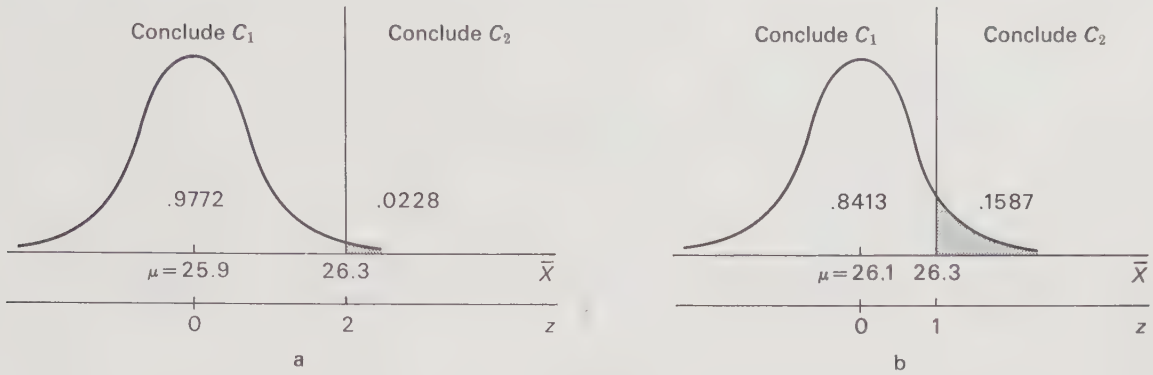



**Figure 16.1**

The conclusion probabilities for any other possible values of $\mu$ are obtained in a similar fashion. For instance, if $\mu = 26.1$ pounds, the sampling distribution of $\bar{X}$ is identical to the one shown in Figure 16.1(a) except it will have a mean equal to 26.1 rather than 25.9. See Figure 16.1(b). In this case, the $z$ value corresponding to $\bar{X} = 26.3$ is $z = (26.3 - 26.1)/.2 = 1$. Thus, from Table A-1, it follows that:

$$P(C_1 : \mu = 26.1) = P(\bar{X} \leqslant 26.3 : \mu = 26.1) = P(z \leqslant 1) = .8413;$$

$$P(C_2 : \mu = 26.1) = P(\bar{X} > 26.3 : \mu = 26.1) = P(z > 1) = .1587.$$

**Problem 6**

Refer to the decision rule given in Example 3, where $n = 49$ and $\sigma = 1.4$.

(a) For $\mu = 26.3$, draw a diagram similar to those in Figure 16.1.√

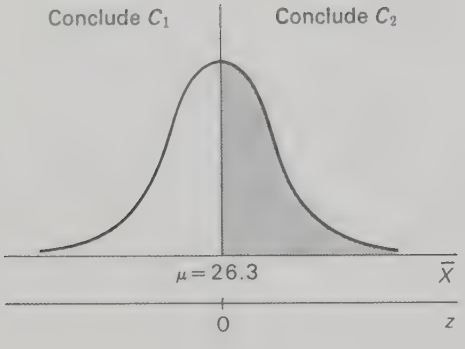
(b) Calculate the conclusion probabilities for the $\mu$ values shown in Table 16.1.

**Table 16.1**

| Mean Weight of Shafts in the Shipment ($\mu$) | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ |
|---|---|---|
| 25.9 | .9772 | .0228 |
| 26.1 | .8413 | .1587 |
| 26.3 | ——— | ——— |
| 26.5 | ——— | ——— |
| 26.7 | ——— | ——— √ |

**Answers**

**6. (a)**




**(b)**

| $\mu$ | $z$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ |
|---|---|---|---|
| 25.9 | +2 | .9772 | .0228 |
| 26.1 | +1 | .8413 | .1587 |
| 26.3 | 0 | .5000 | .5000 |
| 26.5 | –1 | .1587 | .8413 |
| 26.7 | –2 | .0228 | .9772 |

**If you had no difficulty with Problem 6, go on to Topic E: Otherwise do Problem 7.**

**Problem 7**

For Problem 1, assume that management specifies a simple random sample of size 36 to be selected and the following decision rule applied:

$$\text{If } \bar{X} \leqslant 200, \text{ conclude } C_1 \ (\mu \leqslant 200); \quad \text{if } \bar{X} > 200, \text{ conclude } C_2 \ (\mu > 200).$$

Determine the conclusion probabilities for the $\mu$ values shown below, assuming that the population standard deviation is $\sigma = 30$ hours.

| Mean Service Life ($\mu$) | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ |
|---|---|---|
| 190 | _____ | _____ |
| 200 | _____ | _____ |
| 210 | _____ | _____ √ |

**E. Power Curve and Operating Characteristic Curve.** For a given decision rule, the *power curve* shows the probability of concluding $C_2$ under different values of the outcome state $\mu$, while the *operating characteristic curve* shows the probability of concluding $C_1$. Thus:

power curve = $P(C_2 : \mu)$ for all values of $\mu$.

operating characteristic curve = $P(C_1 : \mu)$ for all values of $\mu$.

Example 5 In Problem 6, we obtained five ordinates on the power curve and five ordinates on the operating characteristic curve corresponding to the $\mu$ values, 25.9, 26.1, 26.3, 26.5, and 26.7, for the decision rule: If $\bar{X} \leqslant 26.3$, conclude $C_1$; if $\bar{X} > 26.3$, conclude $C_2$. Sketches of the power curve $P(C_2 : \mu)$ and the operating characteristic curve $P(C_1 : \mu)$ are given in Figure 16.2.

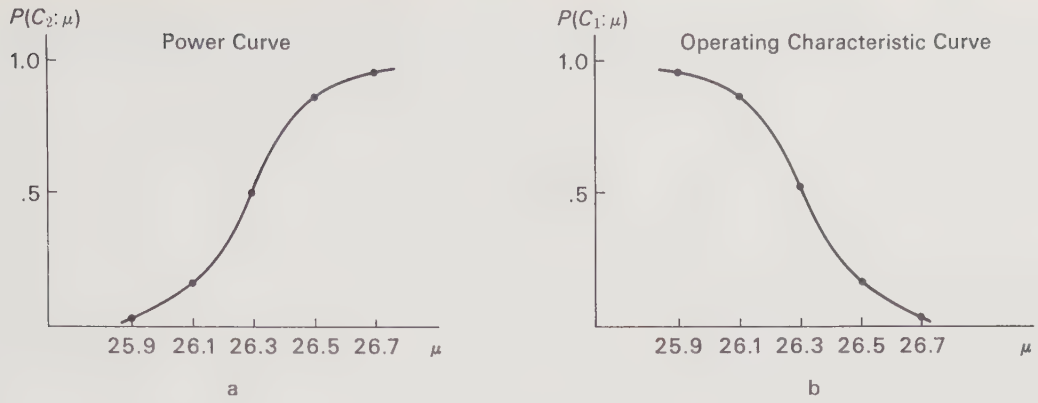



**Figure 16.2**

**Problem 8**

For $\sigma = 1.4$, $n = 49$, and the decision rule: If $\overline{X} \leqslant 26.3$, conclude $C_1$; if $\overline{X} > 26.3$, conclude $C_2$:

(a) Determine the value of the power curve ordinate at $\mu = 26.0$. _____

(b) Determine the value of the operating characteristic curve ordinate at $\mu = 26.0$. _____

_____ √

(c) What is the sum of the two values obtained in (a) and (b) above? _____

(d) Is this sum the same for all values of $\mu$? _____

(e) Therefore, how can one obtain the power curve from the operating characteristic curve, or vice versa?

_____

_____ √

**Problem 9**

Sketch the power curve for the statistical decision rule in Problem 7. (*Hint*: The conclusion probabilities calculated in Problem 7 provide three ordinates of this power curve.) √

**F. Error Probabilities.** The probabilities of making Type I and Type II errors are called error probabilities, and can be obtained from the power and/or the operating characteristic curves.

Example 6 In the machining illustration of Example 5, where the alternatives were: $C_1 : \mu \leqslant 26$; $C_2 : \mu > 26$, the error probabilities are related to the ordinates of the power curve and operating characteristic curve as follows:

$$P(\text{Type I error}) = P(C_2 : \mu) \text{ when } \mu \leqslant 26.0;$$

$$P(\text{Type II error}) = P(C_1 : \mu) \text{ when } \mu > 26.0.$$

Answers

**7.**

| $\mu$ | $z$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ |
|---|---|---|---|
| 190 | +2 | .9772 | .0228 |
| 200 | 0 | .5000 | .5000 |
| 210 | −2 | .0228 | .9772 |

**8.** (a) $\mu = 26.0$; $z = +1.5$; $P(C_2 : \mu) = .0668$.

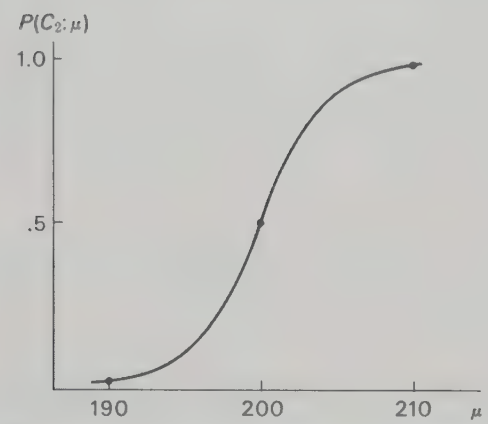
(b) $\mu = 26.0$; $z = +1.5$; $P(C_1 : \mu) = .9332$.

(c) **.0668 + .9332 = 1.**

(d) Yes, $P(C_1 : \mu) + P(C_2 : \mu) = 1$, for all values of $\mu$.

(e) Ordinates on the operating characteristic curve can be derived from the power curve directly by subtracting the power curve ordinate from one, that is, $P(C_1 : \mu) = 1 - P(C_2 : \mu)$. Conversely, the power curve can be derived directly from the operating characteristic curve.

**9.**




(In some books, the $P$(Type I error) is denoted by $\alpha$ and $P$(Type II error) is denoted by $\beta$.) Thus, one can read the error probabilities directly from the power curve and operating characteristic curve as shown by the dark lines in Figure 16.3(a).

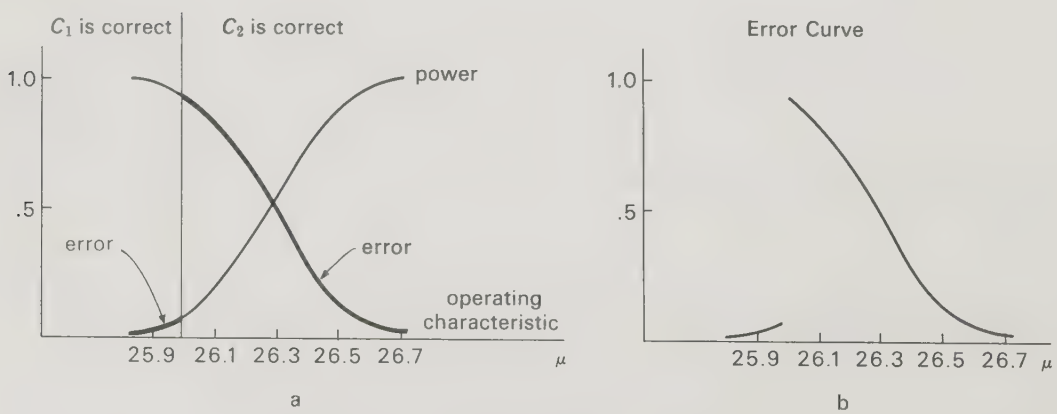



**Figure 16.3**

The resulting error curve is shown in Figure 16.3(b). Note that this error curve shows, for the decision rule under consideration, the probability of making an error for different values of the outcome state $\mu$.

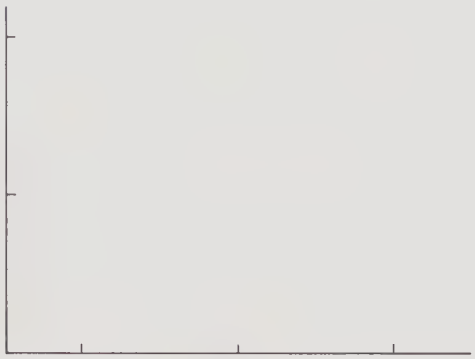
**Problem 10**

Refer to Problem 1.

(a) Fill in the blanks below for this decision problem.

$P$(Type I error) = $P$(____: $\mu$) when $\mu$ _____
$P$(Type II error) = $P$(____ : $\mu$) when $\mu$ _____ √

(b) Sketch the error curve for the statistical decision rule in Problem 7. (*Hint*: Use the sketch of the power curve from Problem 9 to aid you in the construction of this error curve.) √



**Problem 11**

Refer to Example 6.

(a) At what value of $\mu$ is the probability of a Type I error greatest? _____

(b) What is the maximum probability of a Type I error? _____

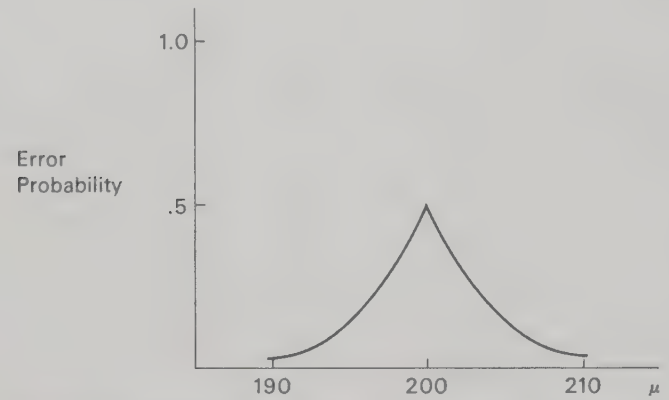
(c) What is the error probability when $\mu$ = 26.5? _____ Is this a Type I or Type II error? _____ √

**Answers**

**10.** **(a)** $P(\text{Type I error}) = P(C_2 : \mu)$ when $\mu \leqslant 200$;

$P(\text{Type II error}) = P(C_1 : \mu)$ when $\mu > 200$.

**(b)**




**Note that the error curve may be obtained from the power curve directly by using the fact that: operating characteristic = 1 − power.**

**11.** **(a)** $\mu = 26.0$.

**(b)** Maximum $P(\text{Type I error}) = P(C_2 : \mu = 26.0) = .0668$.

**(c)** $P(C_1 : \mu = 26.5) = .1587$. Since $C_2$ is the appropriate conclusion when $\mu = 26.5$, it follows that the error must be a Type II error, that is, concluding $C_1$ when $C_2$ is correct.

## REVIEW PROBLEMS

1. Early in 1969, a simple random sample of 2500 families was selected. The change in the annual family income between 1967 and 1968 for each of these families was measured. It was desired to test whether or not the mean change in family income in the population between 1967 and 1968 was positive.

   (a) In this problem, define the population mean $\mu$, and the alternative acts (or conclusions).

   $\mu$: _____

   $C_1$: _____

   $C_2$: _____ √

   (b) In words, define a Type I error and a Type II error in this problem.

   Type I error: _____

   _____

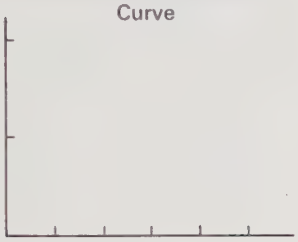
   Type II error: _____

   _____ √

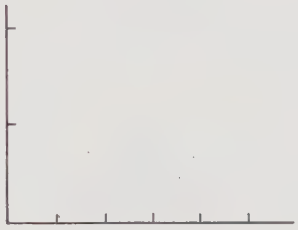
2. Refer to Review Problem 1. At the time the survey was initiated, it was decided that the following decision rule would be used: If $\bar{X} \leqslant \$40$, conclude $C_1(\mu \leqslant 0)$; if $\bar{X} > \$40$, conclude $C_2(\mu > 0)$.

   (a) Construct the power curve, operating characteristic curve, and error curve for the above decision rule. Assume that the population of families is very large relative to the sample size. Use ordinates corresponding to $\mu = 0, +20, +40, +60$, and $+80$ in constructing your curves. Assume that the population standard deviation is $\sigma = \$1000$. √

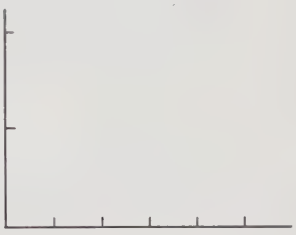
Operating Characteristic Curve



Power Curve



Error Curve



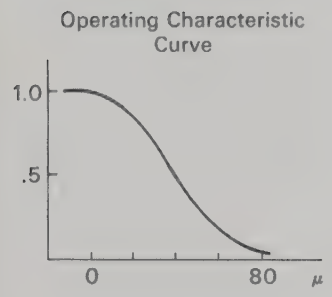
   (b) Suppose a sample of 2500 families had been interviewed and the mean change in income was $\bar{X} = \$80$. What conclusion should be reached with the decision rule above? _____ √

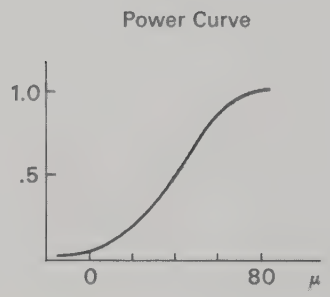
## ANSWERS TO REVIEW PROBLEMS

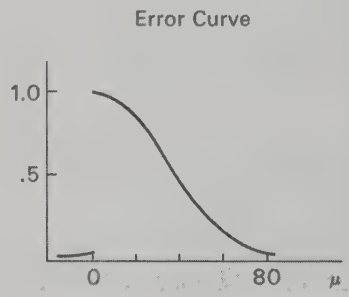
1. (a) $\mu$: population mean change in annual family income between 1967 and 1968. $C_1$: $\mu \leq 0$; $C_2$: $\mu > 0$.
   (b) **Type I error: To conclude that the population mean change in income was positive when it was zero or negative. Type II error: To conclude that the population mean change in income was negative or zero when it was positive.**

2. (a) Sampling distribution of $\bar{X}$: approximately normal, standard deviation = $\sigma/\sqrt{n} = 1000/\sqrt{2500} = 20$.

| $\mu$ | $z = \frac{40 - \mu}{20}$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | $P$(error) |
|---|---|---|---|---|
| 0 | 2 | .9772 | .0228 | .0228 |
| +20 | 1 | .8413 | .1587 | .8413 |
| +40 | 0 | .5000 | .5000 | .5000 |
| +60 | –1 | .1587 | .8413 | .1587 |
| +80 | –2 | .0228 | .9772 | .0228 |




   (b) Since $\bar{X} = 80$ and $80 > 40$, it follows that $C_2$ should be concluded, that is, the population mean change in annual family income between 1967 and 1968 was positive.

# 17

# Construction of Decision Rules Concerning Population Mean – Large Sample

In this section, we consider how to control risks of making incorrect decisions when constructing decision rules. The rules to be taken up concern the population mean and are based on large simple random samples.

**A. Types of Decision Rules.** There are three common types of alternative conclusions involving the population mean. These sets of alternatives (or hypotheses) are listed in Table 17.1. For each type of alternative, the appropriate form of the decision rule is also shown.

**Table 17.1**

| Type | Alternative Conclusions or Hypotheses | Appropriate Form of Decision Rule |
|---|---|---|
| One-sided: upper tail | $C_1$: $\mu \leqslant \mu_0$<br>$C_2$: $\mu > \mu_0$ | If $\bar{X} \leqslant A$, conclude $C_1$;<br>if $\bar{X} > A$, conclude $C_2$ |
| One-sided: lower tail | $C_1$: $\mu \geqslant \mu_0$<br>$C_2$: $\mu < \mu_0$ | If $\bar{X} \geqslant A$, conclude $C_1$;<br>if $\bar{X} < A$, conclude $C_2$ |
| Two-sided | $C_1$: $\mu = \mu_0$<br>$C_2$: $\mu \neq \mu_0$ | If $A_1 \leqslant \bar{X} \leqslant A_2$, conclude $C_1$;<br>if $\bar{X} < A_1$, or $\bar{X} > A_2$,<br>conclude $C_2$ |

Here $\mu_0$ is the indifference level, level of past experience, standard, or the like. The constants $A$, $A_1$, and $A_2$ are *action limits*. Arbitrarily, the alternative containing the equal sign is called $C_1$. For example, the conclusion $\mu \leqslant \mu_0$ is called $C_1$, and $\mu > \mu_0$ is called $C_2$. Frequently, $C_1$ is called the *null hypothesis.*

Example 1 Miracle Manufacturing is considering a new procedure for assembling units. The mean assembly time using the old procedure is 16 minutes. Management wishes to know whether or not the mean assembly time $\mu$ for the new procedure is less than that for the old one.

This is a decision problem of the one-sided, lower tail type, with the alternative conclusions being $C_1$: $\mu \geqslant 16$ minutes, $C_2$: $\mu < 16$ minutes. The appropriate decision rule is of the form: If $\bar{X} \geqslant A$, conclude $C_1$; if $\bar{X} < A$, conclude $C_2$.

**Problem 1**

A food processor receives large shipments of potatoes from producers. If the mean per cent sugar content in the potatoes in any shipment is less than or equal to 15.0, the shipment is of acceptable quality and the regular price

is paid. If the mean content exceeds 15.0, the shipment is considered to be of a lower quality and a lower price is paid to the producer. For this decision problem, identify:

(a) the alternative conclusions $C_1$: _____ $C_2$: _____

(b) the value of $\mu_0$: _____

(c) the appropriate type of decision rule: _____

_____ √

**B. Construction of One-Sided Decision Rule with Control of Type I Error.** When the sample size has been specified, only one of the two types of risks can be controlled, namely through the determination of the action limit(s) of the decision rule. We now consider the construction of a one-sided decision rule when the Type I error is to be controlled.

Example 2 Refer to Example 1, where the alternatives are $C_1$: $\mu \geq 16$, $C_2$: $\mu < 16$. Miracle's management has studied a sample of 64 assembly times using the new procedure. It wishes to construct the decision rule of the form described in Example 1, so as to control the probability of concluding $C_2$, when the mean assembly time for the new procedure is 16 minutes, at the level .01.

This specification indicates that Miracle's management wishes to control a Type I error. Specifically, the Type I error specification has the symbolic form $P(C_2: \mu = 16) = .01$. The level at which the Type I error is to be controlled, .01 here, is often called the *level of significance* of the test.

The appropriate form of the decision rule, as noted above, is: If $\bar{X} \geq A$, conclude $C_1$; if $\bar{X} < A$, conclude $C_2$. Hence, we need to determine the action limit $A$ to satisfy the Type I error specification given above. The determination of $A$ is demonstrated in Figure 17.1.

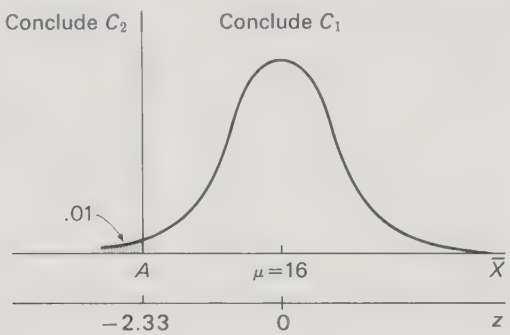



**Figure 17.1**

Since the $z$ value corresponding to a lower tail area of .01 equals −2.33, it follows that $A = 16 - 2.33\,\sigma_{\bar{X}}$.

Ordinarily, one does not know the population standard deviation $\sigma$ and consequently not $\sigma_{\bar{X}}$, the standard deviation of $\bar{X}$. The sample standard deviation $s$ is then used to estimate $\sigma$, and the sample estimate $s_{\bar{X}}$ is used in place of $\sigma_{\bar{X}}$ in the determination of the action limit. For infinite or large finite populations, $s_{\bar{X}} = s/\sqrt{n}$. When $n$ is large, as assumed here, replacing $\sigma_{\bar{X}}$ by $s_{\bar{X}}$ in determining the action limit has no significant effect on the risk control achieved.

Suppose that the following sample results were obtained by Miracle's management:

$$n = 64, \quad \bar{X} = 14.6 \text{ minutes}, \quad s = 4.0 \text{ minutes}.$$

We then have $s_{\bar{X}} = 4.0/\sqrt{64} = .5$, and the action limit which was of the form:

$$A = 16 - 2.33\,\sigma_{\bar{X}}$$

is determined from:

$$A = 16 - 2.33 s_{\bar{X}} = 16 - 2.33(.5) = 14.84.$$

Hence, the decision rule is: If $\overline{X} \geq 14.84$, conclude $C_1 (\mu \geq 16)$; if $\overline{X} < 14.84$, conclude $C_2 (\mu < 16)$. Since $\overline{X} = 14.6$, the conclusion reached is $C_2$, that the mean assembly time for the new procedure is less than 16 minutes.

In some books, the decision rule just employed: If $\overline{X} \geq 16 - 2.33\, s_{\overline{X}}$, conclude $C_1$; if $\overline{X} < 16 - 2.33 s_{\overline{X}}$, conclude $C_2$ is rearranged into the equivalent form: If $(\overline{X} - 16)/s_{\overline{X}} \geq -2.33$, conclude $C_1$; if $(\overline{X} - 16)/s_{\overline{X}} < -2.33$, conclude $C_2$. Note that in this latter form, if $\mu = 16$, $(\overline{X} - 16)/s_{\overline{X}}$ is a type of standardized variable with $s_{\overline{X}}$ replacing $\sigma_{\overline{X}}$.

**Problem 2**

Refer to Problem 1. The food processor takes a simple random sample of 36 potatoes from an incoming shipment. The sample results are $n = 36$, $\overline{X} = 16.4$, and $s = 3.6$.

(a) The food processor wishes to control the probability of concluding $C_2$, when the mean per cent sugar in the potatoes is 15.0, at the .10 level. State this risk specification in symbolic form. _____

(b) Construct a diagram similar to Figure 17.1 to demonstrate the location of the action limit. √

(c) Calculate the action limit. _____

(d) State the complete decision rule. _____

_____ √

(e) What conclusion should be reached? _____

(f) What is the approximate probability of making a Type II error with the decision rule in (d) if $\mu = 16.0$?

_____

(g) Would the risk in (f) be changed if $n$ were increased to 100 and the same decision rule kept? Explain. (No calculations are required here.) _____

_____ √

**C. Construction of One-Sided Decision Rule with Control of Type II Error.**

Example 3 Refer to Example 2. Continue to assume that the sample results were $n = 64$, $\overline{X} = 14.6$, and $s = 4.0$. However, now suppose that Miracle's management wants to set up the decision rule so as to control the probability of concluding $C_1$, when the mean assembly time for the new procedure is 14 minutes, at .025.

Since $C_2$ is the correct conclusion when $\mu = 14$, this is a specification on a Type II error. In symbolic form, it is $P(C_1 : \mu = 14) = .025$. As noted earlier, the appropriate decision rule for the alternatives under consideration is still of the form: If $\overline{X} \geq A$, conclude $C_1$; if $\overline{X} < A$, conclude $C_2$. To determine the action limit $A$ that will satisfy this new risk specification, we employ the diagram in Figure 17.2.

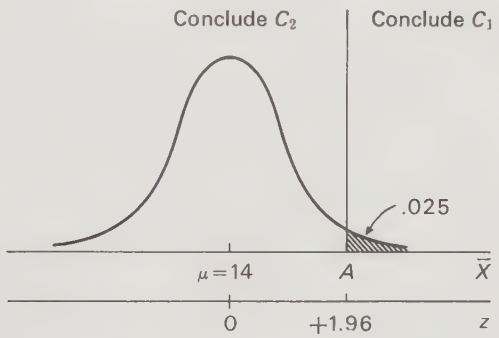



**Figure 17.2**

Answers

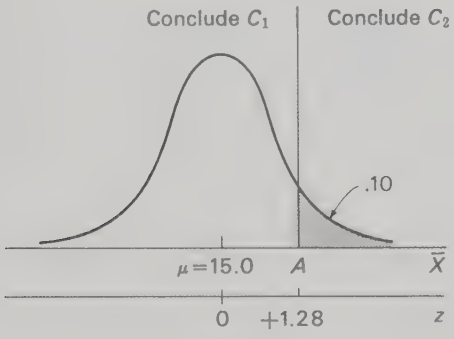
1. (a) $C_1 : \mu \leq 15.0$; $C_2 : \mu > 15.0$.

   (b) $\mu_0 = 15.0$.

   (c) If $\bar{X} \leq A$, conclude $C_1$; if $\bar{X} > A$, conclude $C_2$.

2. (a) $P(C_2 : \mu = 15.0) = .10$.

   (b)




   (c) $s_{\bar{X}} = s/\sqrt{n} = 3.6/\sqrt{36} = .6$ and $z = 1.28$ corresponds to an upper tail area of .10.

   Hence: $A = 15.0 + 1.28(.6) = 15.77$.

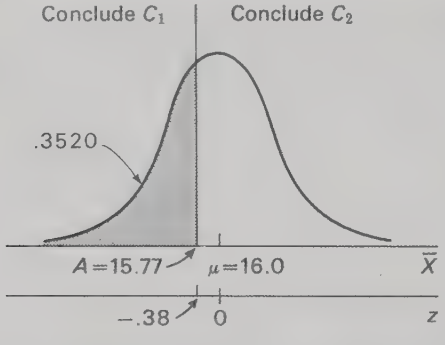
   (d) If $\bar{X} \leq 15.77$, conclude $C_1$; if $\bar{X} > 15.77$, conclude $C_2$.

   The decision rule also may be stated in the following equivalent form:

   If $(\bar{X} - 15.0)/s_{\bar{X}} \leq 1.28$, conclude $C_1$; if $(\bar{X} - 15.0)/s_{\bar{X}} > 1.28$, conclude $C_2$.

   (e) Since $\bar{X} = 16.4 > 15.77$, the decision rule tells us to conclude $C_2$.

   (f) The probability of a Type II error when $\mu = 16.0$ is written symbolically as $P(C_1 : \mu = 16.0)$. The determination of $P(C_1 : \mu = 16.0)$ is demonstrated in the diagram below:




   From (c), we know $A = 15.77$ and $s_{\bar{X}} = .6$. Thus, $z = (15.77 - 16.0)/.6 = -.38$. The lower tail area corresponding to $z = -.38$ is .3520. Hence, $P(C_1 : \mu = 16.0)$ is approximately .3520. This probability is an approximation because the sampling distribution of $\bar{X}$ is only approximately normal (by the central limit theorem) and $s_{\bar{X}}$ was used in place of $\sigma_{\bar{X}}$.

   (g) Yes, because $s_{\bar{X}} = s/\sqrt{n}$ is reduced when $n$ is increased.

Since the $z$ value corresponding to an upper tail area of .025 is +1.96, it follows that $A = 14 + 1.96\sigma_{\bar{X}}$. We now need to estimate $\sigma_{\bar{X}}$. As found in Example 2, we have $s_{\bar{X}} = .5$. Hence the action limit is determined from:

$$A = 14 + 1.96s_{\bar{X}} = 14 + 1.96(.5) = 14.98,$$

and the complete decision rule is: If $\bar{X} \geq 14.98$, conclude $C_1$; if $\bar{X} < 14.98$, conclude $C_2$. Since $\bar{X} = 14.6$, the conclusion to be reached with this decision rule is $C_2$.

**Problem 3**

Refer to Problem 2, where $C_1: \mu \leq 15$, $C_2: \mu > 15$. Continue to assume the sample results are $n = 36$, $\bar{X} = 16.4$, and $s = 3.6$. However, suppose now the food processor wishes to control the probability of concluding $C_1$, when the mean per cent sugar in the potatoes is 16.0, at .005.

(a) State this risk specification in symbolic form. _____

(b) Construct a diagram similar to Figure 17.2 to demonstrate the location of the action limit. √

(c) Calculate the action limit. _____

(d) State the complete decision rule. _____

_____ √

(e) What conclusion should be reached? _____

(f) For what value of $\mu$ is the risk of making a Type I error maximum? _____ √

(g) Estimate the maximum risk of making a Type I error with the decision rule in (d). _____

_____ √

**D. Determination of Sample Size to Control Both Types of Errors—One-Sided Decision Rule.** When the sample size is not predetermined, risks of both Type I and Type II errors can be controlled through the determination of the necessary sample size.

Example 4 Refer to Example 1, where $C_1: \mu \geq 16$, $C_2: \mu < 16$. Suppose now that Miracle's management wishes to control both types of errors, and has not yet decided on a sample size. In particular, assume that the Type I error specification of Example 2, that is, $P(C_2: \mu = 16) = .01$, and the Type II error specification of Example 3, that is, $P(C_1: \mu = 14) = .025$, are both to be satisfied. To determine the necessary sample size to satisfy both risk specifications, we employ Figure 17.3.

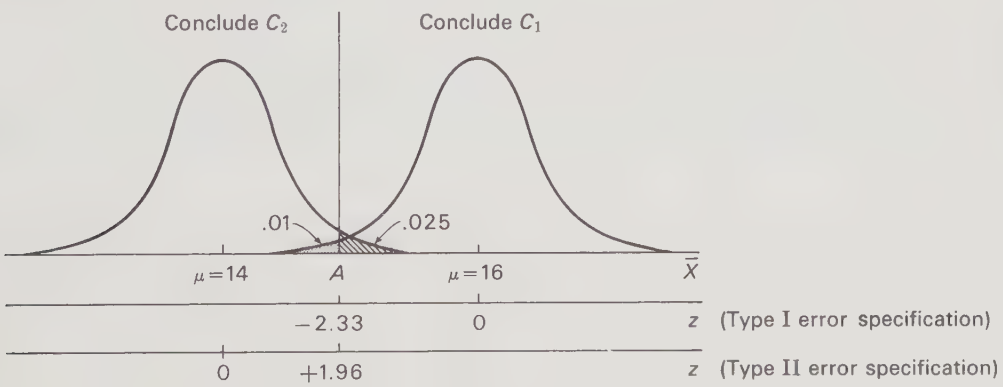
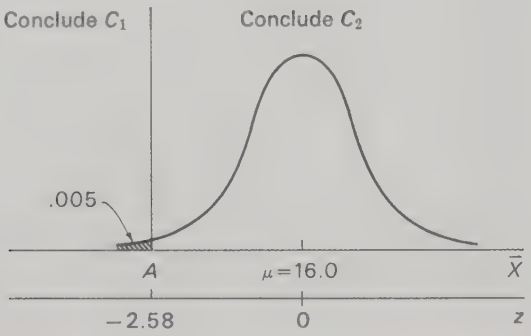



**Figure 17.3**

**Answers**

3. (a) $P(C_1 : \mu = 16.0) = .005$.

(b)




(c) $s_{\bar{X}} = .6$ (see the answer to Problem 2(c)) and $z = -2.58$ corresponds to a lower tail area of .005.

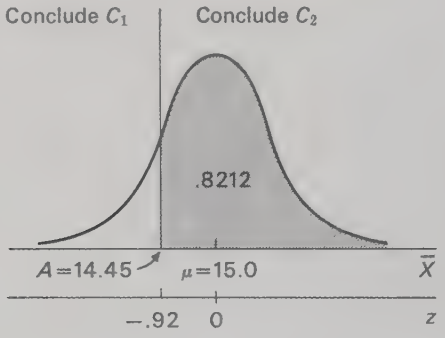
Hence: $A = 16.0 - 2.58(.6) = 14.45$.

(d) If $\bar{X} \leqslant 14.45$, conclude $C_1$; if $\bar{X} > 14.45$, conclude $C_2$.

(e) Since $\bar{X} = 16.4 > 14.45$, the decision rule tells us to conclude $C_2$.

(f) $P(\text{Type I error}) = P(C_2 : \mu)$ when $\mu \leqslant 15.0$. From the decision rule in (d), it follows that the probability of making a Type I error is a maximum when $\mu = 15.0$.

(g) The determination of $P(C_2 : \mu = 15.0)$ is demonstrated in the diagram below:




From (d), we know $A = 14.45$ and $s_{\bar{X}} = .6$. Thus, $z = (14.45 - 15.0)/.6 = -.92$.

The upper tail area corresponding to $z = -.92$ is .8212. Hence, $P(C_2 : \mu = 15.0)$ is approximately .8212.

Generally, the population standard deviation $\sigma$ is unknown, and a planning value for it must be used in order to determine the necessary sample size. Assume that past experience indicates that $\sigma = 4.0$ is a reasonable planning value. Thus $\sigma_{\bar{X}} = 4.0/\sqrt{n}$, where $n$ is the sample size to be determined. The Type I error specification permits us to write the following equation for the action limit $A$:

$$A = 16 - 2.33(4.0/\sqrt{n}).$$

Similarly, the Type II error specification permits us to write a second equation for $A$:

$$A = 14 + 1.96(4.0/\sqrt{n}).$$

Solving the above two equations for $n$ gives:

$$n = \frac{(2.33 + 1.96)^2(4.0)^2}{(16 - 14)^2} = 74.$$

Hence, a sample of 74 assembly times using the new procedure should be selected. The decision rule will be set up after the sample results are obtained using the sample standard deviation $s$, and based on the control of one of the two types of risks of incorrect decisions.

**Problem 4**

Refer to Problem 1. Assume that $\sigma = 3.6$ is a reasonable planning value, that the sample size has not yet been determined, and that the risk specifications to be met are $P(C_1: \mu = 16.0) = .005$ and $P(C_2: \mu = 15.0) = .10$.

(a) Which is the risk specification on the Type II error? _____

(b) Find the necessary sample size that will satisfy both risk specifications. _____ √

(c) Find the necessary sample size if the Type II risk specification referred to $\mu = 15.4$ instead of $\mu = 16.0$, everything else remaining the same. _____

(d) From a comparison of the results in (b) and (c), what generalization is suggested? _____

_____

_____ √

**E. Construction of Two-Sided Decision Rule with Control of Type I Error.**

Example 5 An educator is studying grades predicted by students and their actual grades. Of course, the predictions are made before the actual grades are known. Let $X$ be the difference for a student between his predicted grade and his actual grade. Assume that a simple random sample of 49 students is used, and that the sample results are $n = 49$, $\overline{X} = 4$, and $s = 7$.

Consider the two-sided decision problem involving the alternative conclusions, $C_1: \mu = 0$ and $C_2: \mu \neq 0$, where $\mu$ is the mean prediction error in the population. The appropriate type of decision rule is: If $A_1 \leqslant \overline{X} \leqslant A_2$, conclude $C_1$; if $\overline{X} < A_1$ or $\overline{X} > A_2$, conclude $C_2$. Note that a Type I error can only occur when $\mu = 0$. The two action limits are to be chosen so that they are symmetrically located about zero. Assume that the probability of a Type I error (level of significance) is to be controlled at .003.

The determination of the action limits $A_1$ and $A_2$ to meet the risk specification above is aided by the diagram in Figure 17.4.

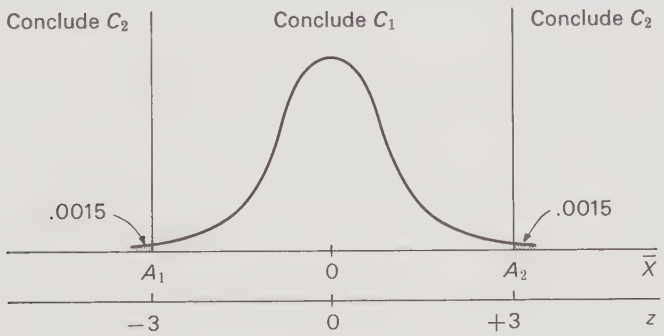


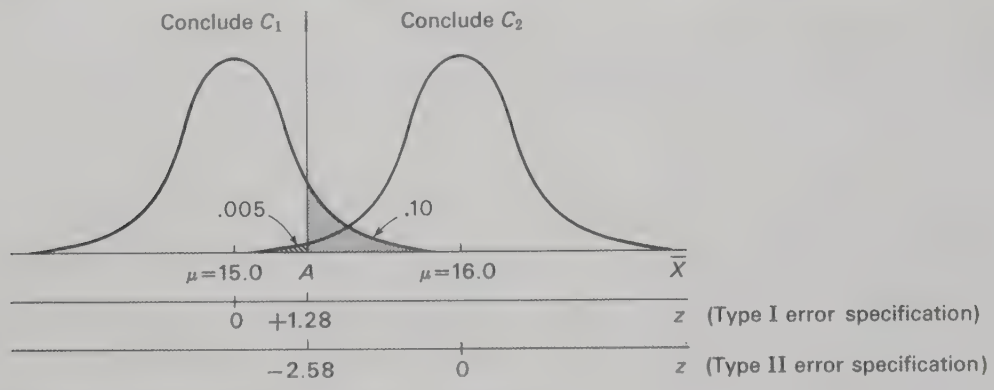

**Figure 17.4**

Answers

4. (a) $P(C_1:\ \mu = 16.0) = .005$.

(b) $\sigma_{\overline{X}} = \sigma/\sqrt{n} = 3.6/\sqrt{n}$.

The determination of the necessary sample size is demonstrated in the diagram below:




The Type I error specification permits us to write:

$$A = 15.0 + 1.28(3.6/\sqrt{n}).$$

The Type II error specification permits us to write:

$$A = 16.0 - 2.58(3.6/\sqrt{n}).$$

Solving for $n$ gives:

$$n = \frac{(1.28 + 2.58)^2\,(3.6)^2}{(15.0 - 16.0)^2} = 193.$$

(c) Replacing $\mu = 16.0$ by $\mu = 15.4$ in the formula for $n$ in (b) gives:

$$n = \frac{(1.28 + 2.58)^2\,(3.6)^2}{(15.0 - 15.4)^2} = 1207.$$

(d) The closer together the two values of the mean between which one wishes to discriminate, the larger the necessary sample size.

Since $z = \pm 3$ corresponds to equal tail areas which sum to .003, it follows that $A_1 = 0 - 3\sigma_{\overline{X}}$ and $A_2 = 0 + 3\sigma_{\overline{X}}$. From the sample results, we obtain an estimate of $\sigma_{\overline{X}}$, namely $s_{\overline{X}} = s/\sqrt{n} = 7/\sqrt{49} = 1$. Hence, we have:

$$A_1 = 0 - 3s_{\overline{X}} = 0 - 3(1) = -3;$$

$$A_2 = 0 + 3s_{\overline{X}} = 0 + 3(1) = +3.$$

The decision rule therefore is: If $-3 \leqslant \overline{X} \leqslant +3$, conclude $C_1$; if $\overline{X} < -3$ or $\overline{X} > +3$, conclude $C_2$. Since $\overline{X} = 4$, we conclude $C_2$.

In some books, the decision rule just employed is rearranged into the equivalent form:

$$\text{If } -3 \leqslant (\overline{X} - 0)/s_{\overline{X}} \leqslant +3, \text{ conclude } C_1; \text{ if } (\overline{X} - 0)/s_{\overline{X}} < -3 \text{ or } (\overline{X} - 0)/s_{\overline{X}} > +3, \text{ conclude } C_2.$$

**Problem 5**

Mean sales per customer for a supermarket averaged \$14.30 prior to the opening of a nearby competitor's store and the simultaneous defensive initiation of a new promotion involving discount coupons. After one month of the new situation, a simple random sample of 256 customers was selected to determine whether or not the mean sales per customer had changed. The sample results were $n = 256$, $\bar{X} = \$13.70$, and $s = \$8.00$.

(a) Specify the alternative conclusions in this problem. $C_1$: _____ ; $C_2$: _____ √

(b) Determine the action limits of the decision rule; assume that the supermarket wants to control the risk of making a Type I error at .05. $A_1$ = _____ ; $A_2$ = _____

(c) State the complete decision rule. _____
_____ √

(d) What conclusion should be drawn? _____

(e) If you were to construct a 95% confidence interval for $\mu$, would it include \$14.30? (*Hint*: Use the result from (d).) _____ √

## REVIEW PROBLEMS

1. Consider the food processor described in Problem 1. The processor and the producers wish to develop an inspection plan for assessing the quality of a potato shipment with respect to sugar content. The alternative conclusions are: $C_1$: $\mu \leqslant 15$, $C_2$: $\mu > 15$. The processor and producers have agreed that the plan should satisfy the following two risk specifications:

$$P(C_1 : \mu = 16) = .01 \text{ and } P(C_2 : \mu = 14) = .01.$$

It is reasonable to work with a planning value of $\sigma = 3.6$.

(a) What sample size should be used in the inspection plan if the above two risk specifications are to be satisfied? $n$ = _____

(b) In this problem, which type of error, Type I or Type II, will penalize the processor? _____ The producers? _____ √

2. A hardware chain wishes to study the effect on profits of handling rental items in addition to its regular hardware line. As a pilot study, rental items are introduced in 36 stores and the changes in the annual profit levels of these stores are recorded. It is desired to test whether or not the average change in the profit level per store is zero. The risk of concluding that the mean change is not zero, when it actually is zero, is to be controlled at .05. Suppose the sample results are $n = 36$, $\bar{X} = \$1540$, and $s = \$8400$.

(a) Define the population mean $\mu$ in this problem. _____
_____

(b) Set up the alternative conclusions for this problem. $C_1$: _____ ; $C_2$: _____ √

(c) Set up the decision rule for this problem. _____
_____

(d) What conclusion should be drawn? _____ √

3. Refer to Review Problem 2. Suppose management had desired to test whether or not the average change in the profit level per store is positive. It wishes to control the probability of concluding that the mean change is positive, when actually no change occured, at .025.

(a) Set up the alternative conclusions for this problem. $C_1$: _____ ; $C_2$: _____ √

(b) Set up the decision rule for this problem. _____
_____

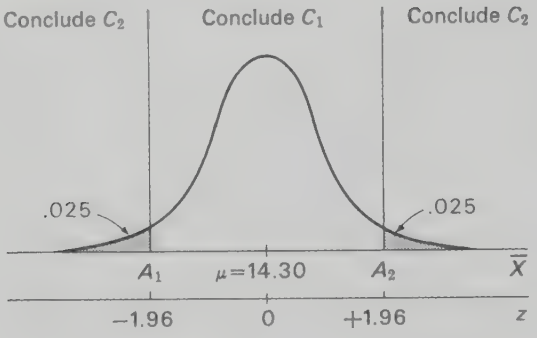
(c) What conclusion should be reached? _____ √

(d) What is the approximate probability of concluding from the decision rule in (b) that the addition of the rental items does not increase profit levels on the average when, in fact, $\mu = +\$5000$? _____
_____ √

Answers

5. (a) $C_1$: $\mu = 14.30$; $C_2$: $\mu \neq 14.30$.

(b)




$s_{\bar{X}} = 8.00/\sqrt{256} = .50$ and $z = \pm 1.96$.

Thus: $A_1 = 14.30 - 1.96(.50) = 13.32$;

$A_2 = 14.30 + 1.96(.50) = 15.28$.

(c) If $13.32 \leqslant \bar{X} \leqslant 15.28$, conclude $C_1$; if $\bar{X} < 13.32$ or $\bar{X} > 15.28$, conclude $C_2$.

The decision rule also may be stated in the following equivalent form:

If $-1.96 \leqslant (\bar{X} - 14.30)/s_{\bar{X}} \leqslant +1.96$, conclude $C_1$; if $(\bar{X} - 14.30)/s_{\bar{X}} < -1.96$ or $(\bar{X} - 14.30)/s_{\bar{X}} > +1.96$, conclude $C_2$.

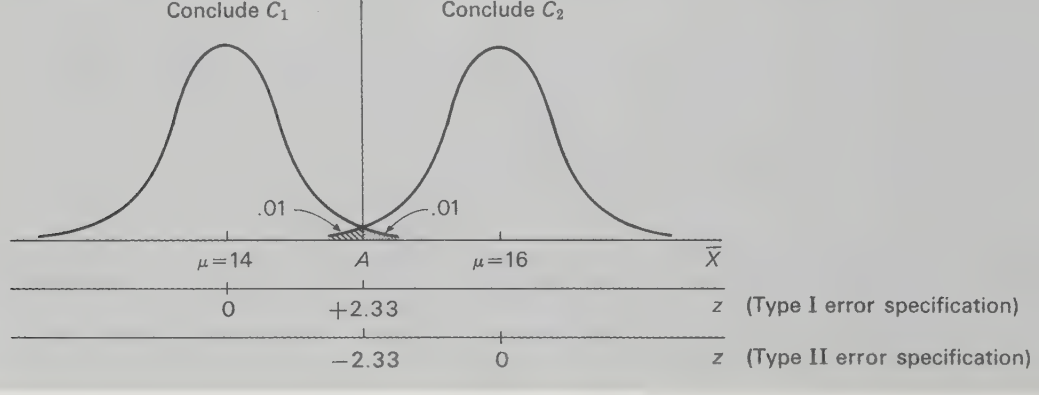
(d) Since $\bar{X} = 13.70$ and $13.32 \leqslant 13.70 \leqslant 15.28$, conclude $C_1$.

(e) Yes. Since $C_1$ ($\mu = 14.30$) is the conclusion drawn in (d), it follows that $\mu = 14.30$ would be contained in a 95% confidence interval for $\mu$ constructed from the sample data of this problem.

## ANSWERS TO REVIEW PROBLEMS

1. (a) Type I error specification $P(C_2: \mu = 14) = .01$;

Type II error specification $P(C_1: \mu = 16) = .01$.



## ANSWERS TO REVIEW PROBLEMS

**1. (a) cont'd.**

$s_{\bar{X}} = 3.6/\sqrt{n}$ and $A = 14 + 2.33(3.6/\sqrt{n})$; $A = 16 - 2.33(3.6/\sqrt{n})$.

Solving for $n$ gives:

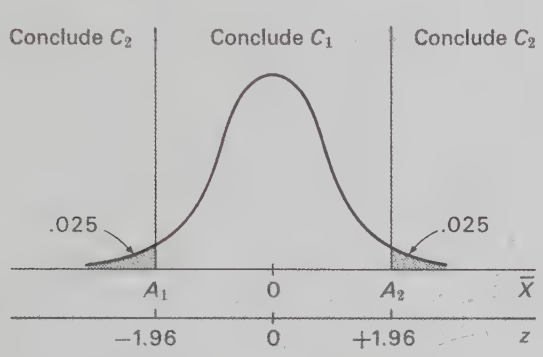
$$n = \frac{(2.33 + 2.33)^2 (3.6)^2}{(14 - 16)^2} = 70.$$

**(b)** A Type II error penalizes the processor. A Type I error penalizes the producers.

**2. (a)** $\mu$: mean change in the annual profit level per store in the hardware chain.

**(b)** $C_1$: $\mu = 0$; $C_2$: $\mu \neq 0$.

**(c)**




$s_{\bar{X}} = 8400/\sqrt{36} = 1400$ and $z = \pm 1.96$.

Thus: $A_1 = 0 - 1.96(1400) = -2744$;

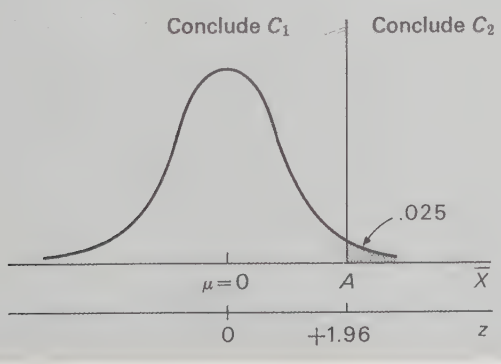
$A_2 = 0 + 1.96(1400) = +2744$.

The decision rule is: If $-2744 \leq \bar{X} \leq +2744$, conclude $C_1$; if $\bar{X} < -2744$ or $\bar{X} > +2744$, conclude $C_2$.

**(d)** Since $\bar{X} = 1540$ and $-2744 \leq 1540 \leq +2744$, conclude $C_1$.

**3. (a)** $C_1$: $\mu \leq 0$; $C_2$: $\mu > 0$.

**(b)**



## ANSWERS TO REVIEW PROBLEMS

3. (b) cont'd.

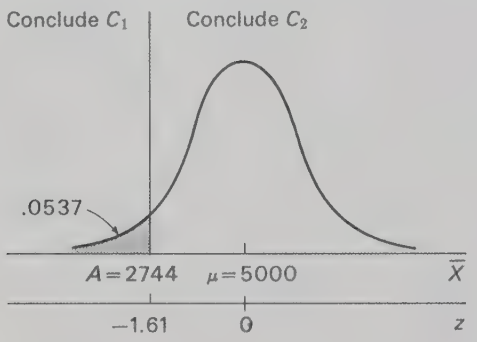
$s_{\bar{X}} = 1400$ (see the answer to Review Problem 2 (c)) and $z = 1.96$;

$A = 0 + 1.96(1400) = 2744.$

The decision rule is: If $\bar{X} \leqslant 2744$, conclude $C_1$; if $\bar{X} > 2744$, conclude $C_2$.

(c) Since $\bar{X} = 1540 \leqslant 2744$, conclude $C_1$.

(d) Stated symbolically, the probability to be determined is $P(C_1 : \mu = +5000)$.




From (b), we know $A = 2744$ and $s_{\bar{X}} = 1400$. Thus, $z = (2744 - 5000)/1400 = -1.61$.

The lower tail area corresponding to $z = -1.61$ is .0537. Hence, $P(C_1 : \mu = +5000)$ is approximately .0537.

# 18

# Construction of Decision Rules Concerning the Difference Between Two Population Means – Large Samples

In this section, we consider the construction of decision rules for the difference between two population means, based on large, independent random samples from the two populations under study. (If you have not yet worked through Topic A in Section 12, you should do so now prior to working this section.)

The decision rules involve $D = \bar{X}_2 - \bar{X}_1$, where $\bar{X}_1$ and $\bar{X}_2$ are the two sample means based on samples of $n_1$ and $n_2$ respectively. The mean and standard deviation of the sampling distribution of $D$ are $\mu_D = \mu_2 - \mu_1$ and $\sigma_D = \sqrt{\sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2}}$. If the populations are infinite or very large, we have $\sigma_D = \sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}$. Finally, when both $n_1$ and $n_2$ are large, as is assumed here, the sampling distribution of $D$ is approximately normal. Thus, the construction of decision rules based on $D$ corresponds to the construction of decision rules based on $\bar{X}$ discussed in Section 17.

Generally $\sigma_D$ must be estimated since the population standard deviations are usually unknown. Using the sample standard deviations $s_1$ and $s_2$, we obtain $s_D = \sqrt{s_1^2/n_1 + s_2^2/n_2}$. When the sample sizes are large, the use of the sample estimate $s_D$ in the construction of the decision rule has no significant effect on the error control achieved.

### A. Construction of Decision Rule.

Example 1 A marketing research consultant obtained two used sports cars of a popular make that were alike in model year and general condition, but differed in color. One hundred car salesmen were selected and divided randomly into two equal groups, one group for each car. A cash offer for the car was solicited from each person in the group assigned to that car. The sample results are:

| Color 1 | Color 2 |
|---|---|
| $n_1 = 50$ | $n_2 = 50$ |
| $\bar{X}_1 = \$2730$ | $\bar{X}_2 = \$2915$ |
| $s_1 = \$230$ | $s_2 = \$265$ |

The consultant now wants to decide whether the color of the car has any effect on the size of the offer, that is, whether or not the mean offer is the same for the two colors. The alternative conclusions for this problem are:

$$C_1: \ \mu_1 = \mu_2, \qquad C_1: \ \mu_2 - \mu_1 = 0,$$
$$\text{or equivalently}$$
$$C_2: \ \mu_1 \neq \mu_2, \qquad C_2: \ \mu_2 - \mu_1 \neq 0.$$

Thus, a two-sided decision rule is appropriate. Suppose the consultant wishes to control at the level .046 the risk of concluding $\mu_1 \neq \mu_2$ when actually $\mu_1 = \mu_2$ (i.e., the level of significance is .046).

The appropriate type of decision rule is: If $A_1 \leq D \leq A_2$, conclude $C_1$; if $D < A_1$ or $D > A_2$, conclude $C_2$. The determination of the two action limits is demonstrated in Figure 18.1.

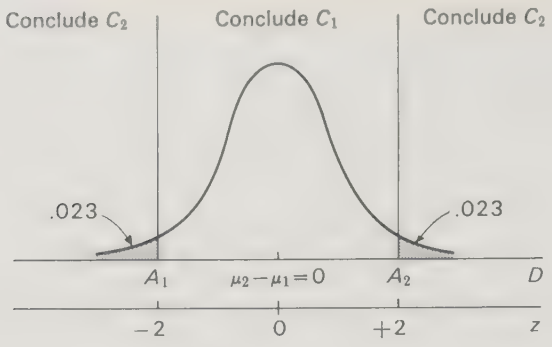



**Figure 18.1**

Since the $z$ values corresponding to a total area of .046 in the two tails are ±2, it follows that $A_1 = 0 - 2\,\sigma_D$ and $A_2 = 0 + 2\,\sigma_D$. We estimate $\sigma_D$ from the sample results as follows: $s_D = \sqrt{(230)^2/50 + (265)^2/50} = 49.6$. The action limits are then obtained as follows:

$$A_1 = 0 - 2s_D = 0 - 2(49.6) = -99.2;$$

$$A_2 = 0 + 2s_D = 0 + 2(49.6) = +99.2.$$

Hence, the decision rule is: If $-99.2 \leqslant D \leqslant +99.2$, conclude $C_1$; if $D < -99.2$ or $D > +99.2$, conclude $C_2$, or equivalently, if $-2 \leqslant (D-0)/s_D \leqslant +2$, conclude $C_1$; if $(D-0)/s_D < -2$ or $(D-0)/s_D > +2$, conclude $C_2$.

Since $D = \bar{X}_2 - \bar{X}_1 = 2915 - 2730 = 185$, we conclude $C_2$, that the mean offers for the two colors are not the same.

**Problem 1**

A trucking company wishes to determine whether or not batteries of type B have a longer mean service life than batteries of type A. An experiment was conducted, in which the service lives of 121 batteries of type A had a mean of 18.4 months and a standard deviation of 3.3 months, while the service lives of 36 batteries of type B had a mean of 21.3 months and a standard deviation of 4.2 months. The experiment was designed so that these two sets of batteries represent independent random samples. Let $\mu_A$ and $\mu_B$ denote the mean service lives of batteries of type A and type B respectively.

(a) State the two alternative conclusions. $C_1$: _____ ; $C_2$: _____ √

(b) Management wishes to control the risk of concluding that $\mu_B$ exceeds $\mu_A$, when actually they are equal, at the level .05. State this risk specification symbolically. _____

(c) Construct a diagram similar to Figure 18.1 to demonstrate the location of the action limit(s). Use the statistic $D = \bar{X}_B - \bar{X}_A$. √

(d) Determine the decision rule. _____

_____

(e) What conclusion should be drawn? _____ √

(f) For the decision rule constructed in (d), determine the approximate probability that conclusion $C_1$ will be reached when $\mu_B - \mu_A = 3$. (*Hint*: What is the value of $\mu_D$ when $\mu_B - \mu_A = 3$?) _____ √

## REVIEW PROBLEMS

1. During 1967, the mean and standard deviation of the number of complaints received each week by a mail order house were 4.8 and 2.0 respectively. On January 1, 1968, a new quality control program was instituted in an attempt to reduce customer complaints. During 1968, the mean and standard deviation of the number of complaints received each week were 3.8 and 2.5 respectively. Assume the 52 sample values in each year represent independent random samples. The company wishes to test whether or not the mean number of complaints per week with the new quality control program ($\mu_2$) is less than the mean number prior to the new program ($\mu_1$).
   (a) Specify the alternative conclusions in this problem. $C_1$: _____; $C_2$: _____ √
   (b) The company desires to control at .01 the probability of concluding there has been a decline, when actually no change in the mean number of complaints per week has resulted. State this risk specification symbolically. _____
   (c) Construct the appropriate decision rule. Use the test statistic $D = \bar{X}_2 - \bar{X}_1$. _____
   (d) What conclusion should be drawn? _____ √
   (e) For the decision rule constructed in (c), what is the approximate probability of concluding $C_1$ when, in fact, $\mu_2 - \mu_1 = -2.0$? _____ √

Answers

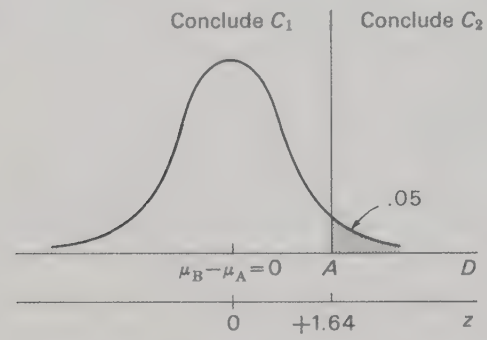
1. (a) $C_1$: $\mu_B \leq \mu_A$, $C_1$: $\mu_B - \mu_A \leq 0$,

or equivalently

$C_2$: $\mu_B > \mu_A$, $C_2$: $\mu_B - \mu_A > 0$.

(b) $P(C_2: \mu_B = \mu_A) = P(C_2: \mu_B - \mu_A = 0) = .05$.

(c)



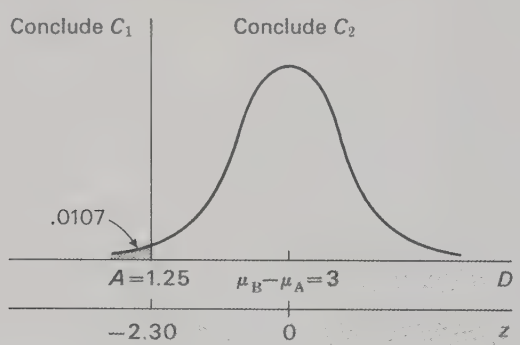

(d) $s_D = \sqrt{(3.3)^2/121 + (4.2)^2/36} = .762$ and $z = 1.64$ corresponds to an upper tail area of .05. Thus, $A = 0 + 1.64(.762) = 1.25$. The decision rule is: If $D \leq 1.25$, conclude $C_1$; if $D > 1.25$, conclude $C_2$.

The decision rule also may be stated in the following equivalent form:

If $(D - 0)/s_D \leq 1.64$, conclude $C_1$; if $(D - 0)/s_D > 1.64$, conclude $C_2$.

(e) Since $D = \bar{X}_B - \bar{X}_A = 21.3 - 18.4 = 2.9$ and $2.9 > 1.25$, we conclude $C_2$.

(f) Stated symbolically, the probability to be determined is: $P(C_1: \mu_B - \mu_A = 3)$.




From (d), we know $A = 1.25$ and $s_D = .762$. Thus, $z = (1.25 - 3)/.762 = -2.30$. The lower tail area corresponding to $z = -2.30$ is .0107. Hence, $P(C_1: \mu_B - \mu_A = 3)$ is approximately .0107.

## ANSWERS TO REVIEW PROBLEMS

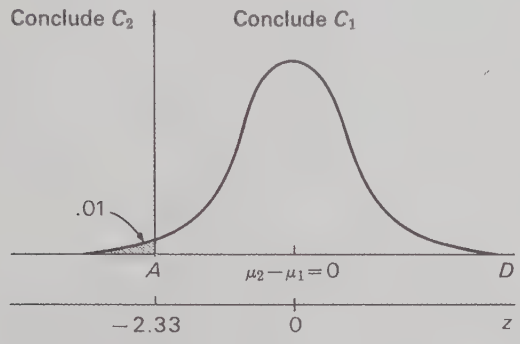
**1.** **(a)** $C_1$: $\mu_2 \geqslant \mu_1$, $\quad C_1$: $\mu_2 - \mu_1 \geqslant 0$,

or equivalently

$C_2$: $\mu_2 < \mu_1$, $\quad C_2$: $\mu_2 - \mu_1 < 0$.

**(b)** $P(C_2\colon\ \mu_2 = \mu_1) = P(C_2\colon\ \mu_2 - \mu_1 = 0) = .01$.

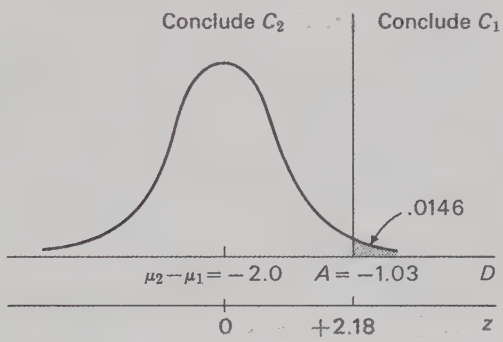
**(c)**




$s_D = \sqrt{(2.0)^2/52 + (2.5)^2/52} = .444$ and $z = -2.33$ corresponds to a lower tail area of .01. Thus, $A = 0 - 2.33(.444) = -1.03$. The decision rule is: If $D \geqslant -1.03$, conclude $C_1$; if $D < -1.03$, conclude $C_2$.

**(d)** Since $D = \bar{X}_2 - \bar{X}_1 = 3.8 - 4.8 = -1.0$ and $-1.0 \geqslant -1.03$, conclude $C_1$.

**(e)** Stated symbolically, the probability to be determined is: $P(C_1\colon\ \mu_2 - \mu_1 = -2)$.




From (c), we know $A = -1.03$ and $s_D = .444$. Thus, $z = (-1.03 + 2.0)/.444 = 2.18$. The upper tail area corresponding to $z = 2.18$ is .0146. Hence, $P(C_1\colon\ \mu_2 - \mu_1 = -2)$ is approximately .0146.

# 19

# Construction of Decision Rules Concerning Population Mean – Normal Population, Small Sample

In this section, we consider the construction of decision rules concerning the mean of a normal population, based on a small random sample.

If $\sigma$ is known, the procedures described in Section 17, using the known $\sigma_{\bar{X}}$, are appropriate regardless of the sample size. If $\sigma$ is unknown so that the sample standard deviations must be used to estimate $\sigma$, the procedures of Section 17 must be modified when the sample size is small. This modification involves the use of the $t$ distribution. (The reader who has not studied the $t$ distribution should refer to Section 13, Topic A.)

**A. Construction of Decision Rule for Population Mean.** When the population is normal, $(\bar{X} - \mu)/s_{\bar{X}}$ follows a $t$ distribution with $n - 1$ degrees of freedom. Here $s_{\bar{X}} = s/\sqrt{n}$. Consequently, the construction of the decision rule will utilize a $t$ value where earlier we used $z$, but otherwise no new problems arise in the construction of decision rules concerning the mean of a normal population.

Example 1 An engineering specification states that the mean service life of a certain machine component must be at least 800 hours. It is desired to test whether this specification is met in the current production process. The decision rule is to be constructed so that the probability of concluding that the mean service life is less than 800 hours, when in fact it is 800 hours, is .05 (i.e., level of significance is .05).

A random sample of four components is selected and the results are $n = 4$, $\bar{X} = 680$ hours, and $s = 100$ hours. Assume that the service lives of these machine components are normally distributed.

The alternative conclusions in this problem are: $C_1$: $\mu \geq 800$, $C_2$: $\mu < 800$. The symbolic form of the error specification is $P(C_2 : \mu = 800) = .05$. The appropriate form of the decision rule is: If $\bar{X} \geq A$, conclude $C_1$; if $\bar{X} < A$, conclude $C_2$. The action limit is found as usual, namely $A = 800 - ts_{\bar{X}}$, except that $t$ has replaced $z$. An analogous picture demonstrating the location of the action limit is found in Section 17, Example 2.

The $t$ value is obtained from the $t$ distribution with $n - 1 = 4 - 1 = 3$ degrees of freedom. Since small values of $\bar{X}$ lead to $C_2$, we require the $t$ value corresponding to a lower tail area of .05 for three degrees of freedom. We use the column headed .10, since the captions in Table A-5 refer to the total area in the two tails, and find $t = -2.353$. We also have $s_{\bar{X}} = 100/\sqrt{4} = 50$, so that we obtain $A = 800 - 2.353(50) = 682.35$. Hence the decision rule is: If $\bar{X} \geq 682.35$, conclude $C_1$; if $\bar{X} < 682.35$, conclude $C_2$. Since $\bar{X} = 680$, we conclude $C_2$, that the mean service life of the machine component is less than 800 hours.

The decision rule also can be expressed in the equivalent form: If $(\bar{X} - 800)/s_{\bar{X}} \geq -2.353$, conclude $C_1$; if $(\bar{X} - 800)/s_{\bar{X}} < -2.353$, conclude $C_2$.

**Problem 1**

A catering firm maintains a fleet of delivery vans from which coffee and other foods are sold to workers at industrial and construction sites. The firm wishes to determine whether or not female van drivers would increase the mean weekly sales revenue per van. In the past, male van drivers have averaged $847 in weekly sales revenue per van. As a pilot study, women drivers were used in four vans. To reduce the possible influence of the novelty, no sales data for the first two weeks of the pilot study are analyzed. Data for the third week are $n = 4$, $\bar{X} = \$1033$,

and $s = \$124$. Assume that weekly van revenues are normally distributed.

(a) What are the alternative conclusions here? $C_1$: _____; $C_2$: _____ √

(b) Assume the firm wishes to control at .05 the probability of concluding that female drivers increase the mean van revenue, when actually the mean is \$847. Construct the appropriate decision rule. _____

_____

(c) What conclusion should be drawn by the firm? _____

(d) What effect, if any, would increasing the sample size have on the action limit? Explain. _____

_____ √

**B. Construction of Decision Rule for Difference between Two Population Means—Equal Variances.** We consider now decision rules for the difference between two population means when independent samples are drawn from each of two normal populations, and both populations have the same variance $\sigma^2$. The sizes of the samples are $n_1$ and $n_2$ respectively.

As usual, $D = \bar{X}_2 - \bar{X}_1$, and the estimated standard deviation of the sampling distribution of $D$ is $s_D = \sqrt{s^2/n_1 + s^2/n_2} = \sqrt{s^2\,(1/n_1 + 1/n_2)}$. Here $s^2$ is the estimate of the common variance $\sigma^2$. This estimate is a pooled variance, being a weighted average of the sample variances $s_1^2$ and $s_2^2$, namely:

$$s^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)\,s_2^2}{n_1 + n_2 - 2}.$$

Since $(D \quad \mu_D)/s_D$ follows a $t$ distribution with $n_1 + n_2 - 2$ degrees of freedom, the construction of decision rules concerning the difference between two population means corresponds to the procedures discussed in Section 18 for large samples, except that a $t$ value is used instead of $z$.

Example 2 An engineer wishes to determine whether the mean strength of a part is the same for two types of synthetic rubber. A sample of 9 pieces of type I rubber and a sample of 16 pieces of type II were analyzed. Suppose the sample results are:

| Type I | Type II |
|---|---|
| $n_1 = 9$ | $n_2 = 16$ |
| $\bar{X}_1 = 13.8$ | $\bar{X}_2 = 15.3$ |
| $s_1 = 3.9$ | $s_2 = 4.4$ |

Assume the distributions of strengths for the two types of rubber are normal with equal variances.

The alternative conclusions are: $C_1$: $\mu_1 = \mu_2$, $C_2$: $\mu_1 \neq \mu_2$, or equivalently, $C_1$: $\mu_2 - \mu_1 = 0$, $C_2$: $\mu_2 - \mu_1 \neq 0$. The construction of the appropriate decision rule proceeds as follows:

$$s^2 = \frac{(9 - 1)(3.9)^2 + (16 - 1)(4.4)^2}{9 + 16 - 2} = 17.92,$$

which gives:

$$s_D = \sqrt{17.92(1/9 + 1/16)} = 1.76.$$

The appropriate decision rule is a two-sided one: If $A_1 \leqslant D \leqslant A_2$, conclude $C_1$; if $D < A_1$ or $D > A_2$, conclude $C_2$. Suppose the engineer specifies that the risk of making a Type I error should be controlled at .05. Thus, we require the $t$ values corresponding to equal tail areas which sum to .05, for $n_1 + n_2 - 2 = 9 + 16 - 2 = 23$ degrees of freedom. They are $t = \pm 2.069$. Therefore the action limits of the decision rule are: $A_1 = 0 - 2.069\,s_D = 0 - 2.069\,(1.76) = -3.64$ and $A_2 = 0 + 2.069\,s_D = 0 + 2.069(1.76) = +3.64$. Hence, the appropriate decision rule is: If $-3.64 \leqslant D \leqslant +3.64$, conclude $C_1$; if $D < -3.64$ or $D > +3.64$, conclude $C_2$.

Since $D = \bar{X}_2 - \bar{X}_1 = 15.3 - 13.8 = 1.5$ and $-3.64 \leqslant 1.5 \leqslant +3.64$, the engineer should conclude $C_1$, that there is no difference in the mean strengths for the two types of rubber.

**Problem 2**

A razor blade manufacturer is studying two types of blades (A, B) to see if the mean number of comfortable shaves obtained from a package of seven blades is the same for each type of blade. Since he is anxious to obtain the results quickly, twenty men who use a safety razor regularly are selected and divided randomly into two groups of ten, one group for each type of blade. Each man is asked to report on the number of comfortable shaves obtained with a package. The reports of the men are:

| Blade A | Blade B |
|---|---|
| $n_A = 10$ | $n_B = 10$ |
| $\bar{X}_A = 46.9$ shaves | $\bar{X}_B = 75.6$ shaves |
| $s_A = 14.0$ shaves | $s_B = 21.0$ shaves |

Assume that the number of shaves obtained from a package is approximately normally distributed, and the population variances are equal.

(a) What are the alternative conclusions in this problem? $C_1$: _____ ; $C_2$: _____ √

(b) The manufacturer wishes to conduct the test so that the probability of a Type I error is held at .01. Construct the appropriate decision rule for this test. Use the test statistic $D = \bar{X}_B - \bar{X}_A$. _____

_____

(c) What conclusion should be drawn? _____ √

Answers

1. (a) $C_1: \mu \leq 847$; $C_2: \mu > 847$.

(b) Risk specification is $P(C_2: \mu = 847) = .05$. The $t$ value for $4 - 1 = 3$ degrees of freedom corresponding to an upper tail area of .05 is +2.353. Since $s = 124$, it follows that $s_{\bar{X}} = 124/\sqrt{4} = 62$. The action limit of the decision rule is $A = 847 + 2.353(62) = 993$. The appropriate decision rule is: If $\bar{X} \leq 993$, conclude $C_1$; if $\bar{X} > 993$, conclude $C_2$.
The decision rule also may be stated in the following equivalent form:
If $(\bar{X} - 847)/s_{\bar{X}} \leq 2.353$, conclude $C_1$; if $(\bar{X} - 847)/s_{\bar{X}} > 2.353$, conclude $C_2$.

(c) Since $\bar{X} = 1033 > 993$, conclude $C_2$.

(d) The action limit would change because both the $t$ value and $s_{\bar{X}}$ would change.

2. (a) $C_1: \mu_A = \mu_B$, $C_1: \mu_B - \mu_A = 0$,
or equivalently
$C_2: \mu_A \neq \mu_B$, $C_2: \mu_B - \mu_A \neq 0$.

(b) Risk specification is $P(C_2: \mu_A = \mu_B) = P(C_2: \mu_B - \mu_A = 0) = .01$.

$$s^2 = \frac{(10 - 1)(14.0)^2 + (10 - 1)(21.0)^2}{10 + 10 - 2} = 318.5;$$

$$s_D = \sqrt{318.5(1/10 + 1/10)} = 7.98;$$

$t = \pm 2.878$ for a total tail area of .01 and $10 + 10 - 2 = 18$ degrees of freedom.

Thus: $A_1 = 0 - 2.878(7.98) = -23.0$ and $A_2 = 0 + 2.878(7.98) = +23.0$.

The decision rule is: If $-23.0 \leq D \leq +23.0$, conclude $C_1$; if $D < -23.0$ or $D > +23.0$, conclude $C_2$.

(c) Since $D = \bar{X}_B - \bar{X}_A = 75.6 - 46.9 = 28.7$ and $28.7 > 23.0$, conclude $C_2$.

## REVIEW PROBLEMS

1. A gas utility company determined from theoretical considerations that the mean annual gas consumption for a gas furnace, gas water heater, and gas cooking range in one-family residences in its territory should be 180 thousand cubic feet. To test this hypothesis, the company has installed special meters in a random sample of twenty-five residences.The sample results, in thousand cubic feet, are $n = 25$, $\overline{X} = 115$, and $s = 45$. Assume that the annual gas consumption rates are normally distributed.
   (a) Set up the alternative conclusions for this problem. $C_1$: _____ ; $C_2$: _____ √
   (b) Assume the company wishes to control the probability of concluding $C_2$ when $C_1$ is correct at .02. Construct the appropriate decision rule. _____
   _____
   (c) What conclusion should be drawn? _____ √

2. A firm wishes to compare the effectiveness of two different advertising displays. A random sample of twelve stores is selected to use the first display, and independently a second random sample of nine stores to use the second display. For each store, the increase in sales of the company's product during the test period is measured. The firm wishes to test whether or not the mean change in sales per store is the same for each display. Assume that the two populations are normal with equal population variances.
   Suppose the sample results are:

| Display One | Display Two |
|---|---|
| $n_1 = 12$ | $n_2 = 9$ |
| $\overline{X}_1 = \$33$ | $\overline{X}_2 = \$41$ |
| $s_1 = \$10$ | $s_2 = \$15$ |

   (a) What are the alternative conclusions for this problem? $C_1$: _____ ; $C_2$: _____ √
   (b) Construct the decision rule. Use $D = \overline{X}_2 - \overline{X}_1$, and assume that the risk of making a Type I error is to be controlled at .10. _____
   (c) On the basis of the sample results, what conclusion should be drawn? _____
   (d) If the stores are selected purposely so that all the stores using the first display are in one geographical area and all the stores using the second display are in another geographical area, would this fact affect the conclusion drawn from the test? _____
   _____
   _____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a) $C_1: \mu = 180$; $C_2: \mu \neq 180$.

   (b) Risk specification is $P(C_2: \mu = 180) = .02$. $t = \pm 2.492$ for a total tail area of .02 and $25 - 1 = 24$ degrees of freedom. $s_{\bar{X}} = 45/\sqrt{25} = 9$.

   Thus: $A_1 = 180 - 2.492(9) = 157.6$ and $A_2 = 180 + 2.492(9) = 202.4$.

   The decision rule is: If $157.6 \leqslant \bar{X} \leqslant 202.4$, conclude $C_1$; if $\bar{X} < 157.6$ or $\bar{X} > 202.4$, conclude $C_2$.

   (c) Since $\bar{X} = 115 < 157.6$, conclude $C_2$.

2. (a) $C_1: \mu_1 = \mu_2$, $\quad C_1: \mu_2 - \mu_1 = 0$,

   or equivalently

   $C_2: \mu_1 \neq \mu_2$, $\quad C_2: \mu_2 - \mu_1 \neq 0$.

   (b) Risk specification is $P(C_2: \mu_1 = \mu_2) = P(C_2: \mu_2 - \mu_1 = 0) = .10$.

   $$s^2 = \frac{(12-1)(10)^2 + (9-1)(15)^2}{12+9-2} = 152.6;$$

   $$s_D = \sqrt{152.6(1/12 + 1/9)} = 5.45;$$

   $t = \pm 1.729$ for a total tail area of .10 and $12 + 9 - 2 = 19$ degrees of freedom.

   Thus: $A_1 = 0 - 1.729(5.45) = -9.42$ and $A_2 = 0 + 1.729(5.45) = +9.42$.

   The decision rule is: If $-9.42 \leqslant D \leqslant +9.42$, conclude $C_1$; if $D < -9.42$ or $D > +9.42$, conclude $C_2$.

   (c) Since $D = \bar{X}_2 - \bar{X}_1 = 41 - 33 = 8$ and $-9.42 \leqslant 8 \leqslant +9.42$, conclude $C_1$.

   (d) **Yes. Now there no longer would be an unambiguous indication of the effects of the displays per se since the effects of different displays and effects of different geographical areas on changes in sales would be mixed together.**

# 20

# Construction of Decision Rules Concerning Population Proportion – Large Sample

In this section, we consider the construction of decision rules for the population proportion, based on a large random sample. Decision rules for the population proportion $p$ involve the sample proportion $\bar{p}$. The mean and standard deviation of the sampling distribution of $\bar{p}$ are $\mu_{\bar{p}} = p$ and $\sigma_{\bar{p}} = \sqrt{p(1-p)/n}$ respectively. The formula for $\sigma_{\bar{p}}$ is appropriate when the population is infinite or very large. Finally, it is known that the sampling distribution of $\bar{p}$ is approximately normal if the sample size is reasonably large, which is the case considered here.

**A. Power Curve of Decision Rule.** We illustrate now the construction of the power curve of a decision rule involving $\bar{p}$.

Example 1 A firm has prepared a mail solicitation to sell its product. A response rate greater than .05 is required for the solicitation to be profitable. As a pilot test, 400 of the solicitations are to be sent to names selected at random from the mailing list, which is very large.

The alternative conclusions in this problem are: $C_1: p \leqslant .05$, $C_2: p > .05$. Suppose that the following decision rule has been suggested: If $\bar{p} \leqslant .04$, conclude $C_1$; if $\bar{p} > .04$, conclude $C_2$. To construct the power curve for this decision rule, we determine, for different possible values of the population proportion $p$, the probability that the decision rule leads to $C_2 (p > .05)$. Figure 20.1 illustrates the calculation for $p = .03$.

$p = .03$; $\sigma_{\bar{p}} = \sqrt{(.03)(.97)/400} = .00853$;

$z = (.04 - .03)/.00853 = 1.17$.

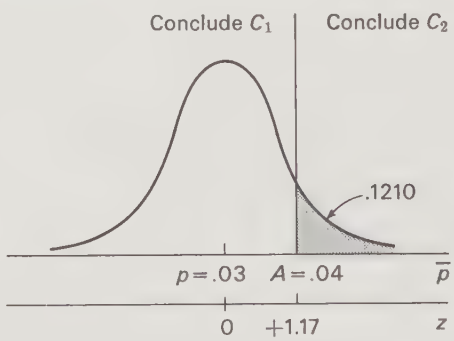



**Figure 20.1**

The area between the mean and .04 is .3790. Hence the ordinate of the power curve at $p = .03$ is .5000 – .3790 = .1210.

**Problem 1**

Refer to Example 1.

(a) If $p = .06$, what is the value of $\sigma_{\bar{p}}$? (Remember, $n = 400$.) _____ √

(For answer, see page 162.)

(b) Why is the value of $\sigma_{\bar{p}}$ in (a) different from that obtained in Example 1? ($\sigma_{\bar{p}} = .00853$ there.)
_____√

(c) Determine the ordinate of the power curve at $p = .06$. Use a diagram such as that shown in Figure 20.1 to illustrate your calculation. _____√

(d) Determine the ordinates of the power curve at $p = .02$ and .04, and sketch the power curve for this decision rule. √

| $p$ | $P(C_2 : p)$ |
|---|---|
| .02 | _____ |
| .03 | .1210 |
| .04 | _____ |
| .06 | _____ see (c) |

(Power Curve)

(e) What are the two types of errors in this decision problem? _____
_____√

(f) Using your results in (d), what is:
(i) the ordinate of the operating characteristic curve at $p = .04$? _____
(ii) the ordinate of the error curve at $p = .04$? _____
(iii) The ordinate of the error curve at $p = .06$? _____√

(g) If it is important not to proceed with the mail solicitation if $p = .04$, is the suggested decision rule an effective one? _____
_____
_____√

**B. Construction of Decision Rule to Control Type I Error.**

Example 2 Refer to Example 1. We will discard the decision rule used there. Suppose the firm wishes to construct a decision rule which will control the probability of concluding $C_2$, when $p = .05$, at the level .05 (i.e., level of significance is .05). The appropriate decision rule is a one-sided, upper tail one. The calculation of the action limit $A$ is illustrated in Figure 20.2.

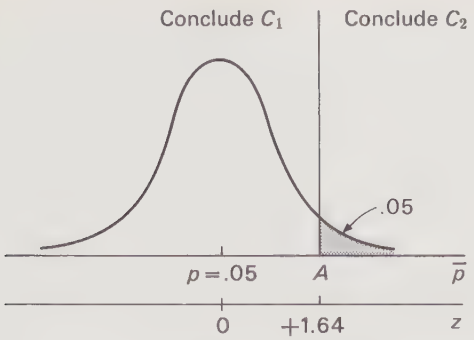



**Figure 20.2**

From the diagram, we see that $A = p + z\,\sigma_{\bar{p}}$ where $p = .05$, $\sigma_{\bar{p}} = \sqrt{(.05)(.95)/400} = .0109$, and $z = 1.64$ corresponds to the .05 shaded area in the upper tail. Substituting, we obtain $A = .05 + 1.64(.0109) = .068$. Thus, the appropriate decision rule is: If $\bar{p} \leqslant .068$, conclude $C_1$; if $\bar{p} > .068$, conclude $C_2$.

The decision rule also can be expressed in the equivalent form: If $(\bar{p} - .05)/.0109 \leqslant 1.64$, conclude $C_1$; if $(\bar{p} - .05)/.0109 > 1.64$, conclude $C_2$.

**Problem 2**

Refer to Example 2.

(a) For decision rules involving $\mu$, we require sample information to calculate $s_{\bar{X}}$ for the decision rule. Why could the decision rule for $p$ in Example 2 be completed prior to obtaining any sample information?

_____

_____√

(b) Suppose that $\bar{p} = .065$ for the sample selected. Using the decision rule in Example 2, what conclusion should be reached?_____

(c) What is the power of the decision rule employed if $p = .10$?_____√

**Problem 3** (For answers, see page 164.)

A firm is undertaking a marketing study to determine whether the proportion of households which use its product currently is larger than .14, the proportion of users in the past. The firm intends to select a simple random sample of 900 households.

(a) What are the alternative conclusions in this decision problem? $C_1$: _____; $C_2$: _____√

(b) What constitute the two types of errors in this problem? _____

_____√

(c) The firm wishes to control at .10 the probability of concluding that the proportion is larger than in the past when actually it has remained unchanged at .14. State the risk specification symbolically. _____

_____√

(d) Construct the appropriate decision rule. _____

_____

(e) In the sample of 900 households, 135 are found to use the firm's product. What conclusion should be reached?_____√

(f) With the decision rule obtained in (d), what is the probability that the firm will conclude the proportion has increased from the past experience when actually it has declined to .12?_____√

### C. Determination of Sample Size to Control Both Types of Errors.

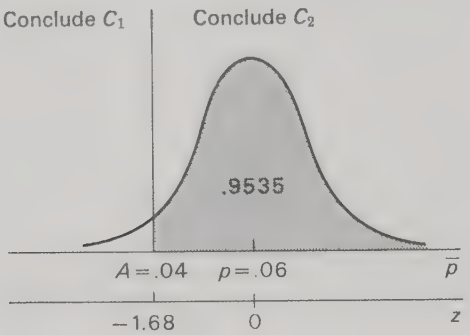
Example 3 Refer to Example 2. Suppose the firm wishes to control both types of risks of incorrect decisions and has not predetermined the size of the sample. The firm wants to control the probability of concluding $C_2$, when $p = .05$, at the level .05. In addition, the firm wants to control the probability of concluding $C_1$, when $p = .07$, at the level .10. Figure 20.3 illustrates the two risk specifications.

**Answers**

**1.** **(a)** $\sigma_{\bar{p}} = \sqrt{(.06)(.94)/400} = .01187.$

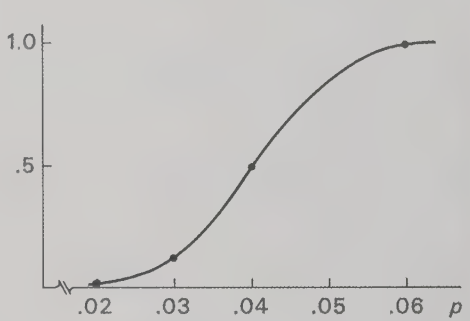
**(b)** The value of $\sigma_{\bar{p}}$ is different because the value of $p$ is different.

**(c)** $z = (.04 - .06)/.01187 = -1.68$. Hence, the ordinate of the curve at $p = .06$ is approximately .9535.




**(d)**

| $p$ | $\sigma_{\bar{p}}$ | $z$ | $P(C_2 : p)$ |
|---|---|---|---|
| .02 | .00700 | 2.86 | .0021 |
| .03 | .00853 | 1.17 | .1210 |
| .04 | .00980 | 0 | .5000 |
| .06 | .01187 | –1.68 | .9535 |




**(e)** Type I error: conclude $C_2$ when $p \leqslant .05$;
Type II error: conclude $C_1$ when $p > .05$.

**(f)** **(i)** .5000;
**(ii)** .5000;
**(iii)** $1.0 - .9535 = .0465$.

**(g)** **Ordinarily the rule would not be considered an effective one, since there is a large probability (.5) of making an incorrect decision which it is important to avoid.**

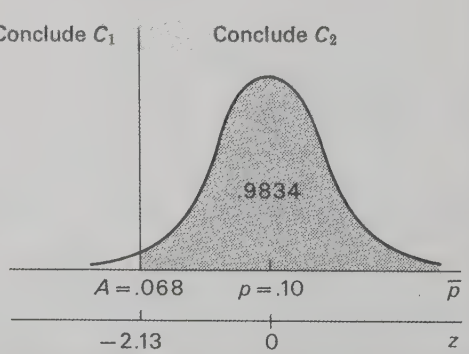
**2.** **(a)** The value of $p$ is specified in the risk specification for constructing the decision rule. Since the value of $\sigma_{\bar{p}}$ depends on the value of $p$, once $p$ is specified, so in turn $\sigma_{\bar{p}}$ is specified.

**(b)** Since $\bar{p} = .065 \leqslant .068$, conclude $C_1$.

**(c)** $p = .10$; $\sigma_{\bar{p}} = \sqrt{(.10)(.90)/400} = .015$;

$z = (.068 - .10)/.015 = -2.13.$

The power of the decision rule is approximately .9834.



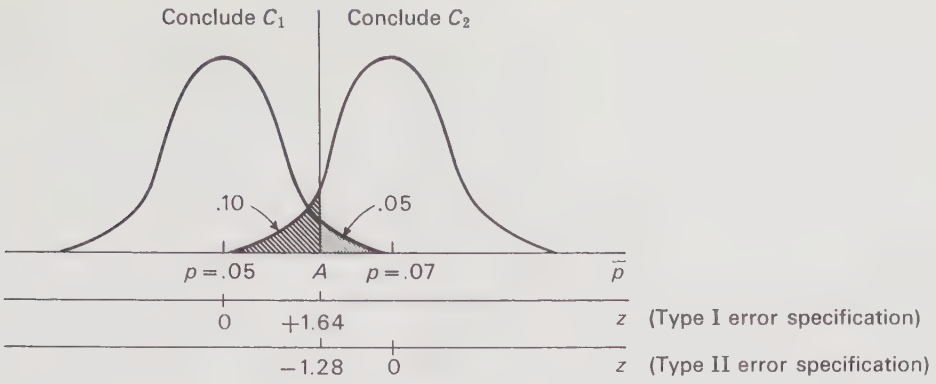



**Figure 20.3**

To obtain the sample size $n$ and action limit $A$ that will satisfy the two risk specifications, we set up two expressions for the action limit $A$, one from each of the risk specifications:

$$A = .05 + 1.64\sqrt{(.05)(.95)/n}\,;$$

$$A = .07 - 1.28\sqrt{(.07)(.93)/n}\,.$$

The two simultaneous equations are now solved for $n$, yielding:

$$n = \frac{[1.64\sqrt{(.05)(.95)} + 1.28\sqrt{(.07)(.93)}]^2}{(.05 - .07)^2} = 1169.$$

Substituting this value into either of the two equations for $A$, we obtain the value of the action limit:

$$A = .05 + 1.64\sqrt{(.05)(.95)/1169} = .060.$$

Thus the firm should send out 1169 solicitations and apply the following decision rule: If $\bar{p} \leqslant .060$, conclude $C_1$; If $\bar{p} > .060$, conclude $C_2$.

Note that the complete decision rule can be set up in advance of sampling when the population proportion is involved, since the standard deviation of the sampling distribution of $\bar{p}$ is known as soon as the value of $p$ is specified for the risk to be controlled.

**Problem 4**

Refer to Problem 3. Assume the firm wishes to control at .10 the probability of concluding that the proportion is larger than .14 when actually it has remained unchanged at .14. Also, the firm wishes to control at .10 the probability of concluding that the proportion is no larger than in the past, when actually it is .16.

(a) Determine the sample size and decision rule which the firm should use. $n =$_____;
decision rule: _____ √

(b) Suppose the sample size obtained in (a) is used and it is found that 228 households in the sample use the firm's product. What conclusion should be drawn? _____ √ (For answer, see page 166.)

**D. Construction of Decision Rule Concerning Difference between Two Population Proportions.** Decision rules concerning the difference between two population proportions involve the statistic $d = \bar{p}_2 - \bar{p}_1$ where $\bar{p}_1$ and $\bar{p}_2$ are the sample proportions of independent large samples of sizes $n_1$ and $n_2$ respectively, drawn from the two populations under study. The mean and standard deviation of the sampling distribution of $d$ are $\mu_d = p_2 - p_1$ and $\sigma_d = \sqrt{\sigma^2_{\bar{p}_1} + \sigma^2_{\bar{p}_2}}$, where $\sigma_{\bar{p}_1}$ and $\sigma_{\bar{p}_2}$ are the standard deviations of the sample proportions. If the populations are infinite or very large, $\sigma^2_{\bar{p}_1} = p_1(1 - p_1)/n_1$ and $\sigma^2_{\bar{p}_2} = p_2(1 - p_2)/n_2$.

Answers

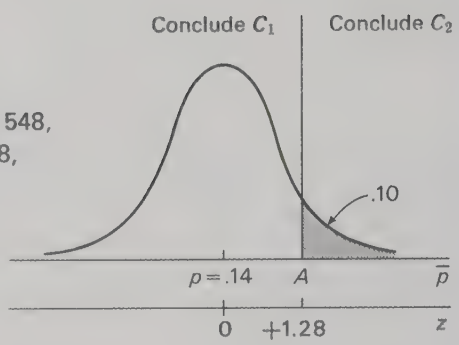
3. (a) $C_1: p \leq .14$; $C_2: p > .14$.

(b) Type I error: conclude $C_2$ when $p \leq .14$;
Type II error: conclude $C_1$ when $p > .14$.

(c) $P(C_2: p = .14) = .10$

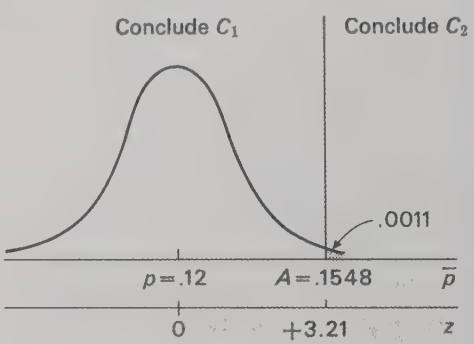
(d) $p = .14$; $\sigma_{\bar{p}} = \sqrt{(.14)(.86)/900} = .01157$;
$A = .14 + 1.28(.01157) = .1548$.

The decision rule is: If $\bar{p} \leq .1548$, conclude $C_1$; if $\bar{p} > .1548$, conclude $C_2$; or equivalently: If $(\bar{p} - .14)/.01157 \leq 1.28$, conclude $C_1$; if $(\bar{p} - .14)/.01157 > 1.28$, conclude $C_2$.

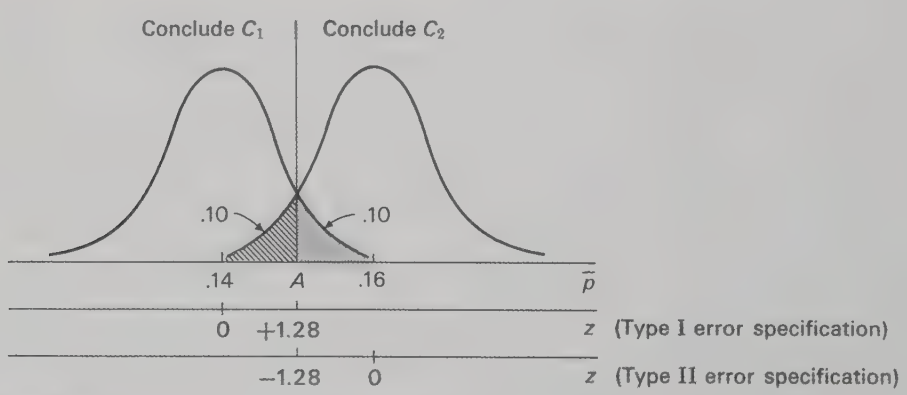



(e) Since $\bar{p} = 135/900 = .150 \leq .1548$, conclude $C_1$.

(f) $p = .12$; $\sigma_{\bar{p}} = \sqrt{(.12)(.88)/900} = .01083$;
$z = (.1548 - .12)/.01083 = 3.21$.
The probability of concluding $C_2$ when $p = .12$ is approximately .0011.




4. (a) Risk specifications: $P(C_2: p = .14) = .10$ and $P(C_1: p = .16) = .10$




(Continued on page 166)

Example 4 A life insurance company makes loans to firms in two geographical regions. Of the 1000 loans made in region A, 18 have resulted in default. Of the 400 loans made in region B, 12 have resulted in default. Assume that these two groups of loans represent independent random samples from the infinite populations of all possible loans which may be made by the company. The insurance company wishes to test whether or not the proportions of loans which result in default are the same in the two populations. The company wishes to hold at .01 the probability of concluding that they differ, when in fact the two proportions are the same.

The alternative hypotheses are: $C_1: p_1 = p_2$, $C_2: p_1 \neq p_2$, or equivalently, $C_1: p_2 - p_1 = 0$, $C_2: p_2 - p_1 \neq 0$. The appropriate decision rule is a two-sided one. To calculate the action limits, we must consider the sampling distribution of $d$ for the case where $p_1 = p_2 = p$; here $p$ is the common value of $p_1$ and $p_2$. In this case, $\sigma_d = \sqrt{p(1-p)/n_1 + p(1-p)/n_2}$. The proportion $p$ is unknown, but may be estimated from the samples by the pooled estimate $\bar{p}$, where $\bar{p} = (n_1\bar{p}_1 + n_2\bar{p}_2)/(n_1 + n_2)$. Substituting $\bar{p}$ for $p$ in $\sigma_d$ gives:

$$s_d = \sqrt{\frac{\bar{p}(1-\bar{p})}{n_1 - 1} + \frac{\bar{p}(1-\bar{p})}{n_2 - 1}},$$

where $s_d$ is the estimator of $\sigma_d$ (In some books $n_1$ and $n_2$ are used instead of $n_1 - 1$ and $n_2 - 1$ in the denominators. When the sample sizes are large, as assumed here, either formula yields practically the same value of $s_d$.)

For the sample data of this problem, $\bar{p}_1 = 18/1000 = .018$ and $\bar{p}_2 = 12/400 = .030$. Thus:

$$\bar{p} = \frac{1000(.018) + 400(.03)}{1000 + 400} = .0214$$

$$\text{and } s_d = \sqrt{\frac{(.0214)(.9786)}{1000 - 1} + \frac{(.0214)(.9786)}{400 - 1}} = .00857.$$

Figure 20.4 shows the sampling distribution of $d = \bar{p}_2 - \bar{p}_1$ when $p_1 = p_2$, together with the location of the action limits.

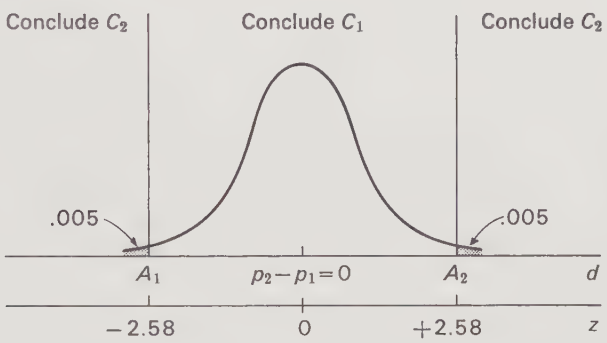



**Figure 20.4**

The action limits, $A_1$ and $A_2$, are equidistant from the center of the distribution. Hence, for a total tail area of .01, $z = \pm 2.58$. Therefore the action limits are $A_1 = 0 - 2.58s_d = 0 - 2.58(.00857) = -.022$ and $A_2 = 0 + 2.58s_d = 0 + 2.58(.00857) = +.022$. Hence, the decision rule is: If $-.022 \leqslant d \leqslant +.022$, conclude $C_1$; if $d < -.022$ or $d > +.022$, conclude $C_2$.

The decision rule also can be expressed in the equivalent form: If $-2.58 \leqslant (d - 0)/.00857 \leqslant +2.58$, conclude $C_1$; if $(d - 0)/.00857 < -2.58$ or $(d - 0)/.00857 > +2.58$, conclude $C_2$.

In this example, $d = \bar{p}_2 - \bar{p}_1 = .030 - .018 = .012$. Therefore, the insurance company should conclude $C_1$, that the proportions of loans which result in default are the same in the two populations.

**Problem 5**

A manufacturer has sold two lots of product, made under identical production conditions, under different warranty contracts. The first lot, containing 5000 units ($n_1$), required 545 warranty adjustments. The second

**Answers**

4. (a) cont'd.

$$A = .14 + 1.28\sqrt{(.14)(.86)/n};$$
$$A = .16 - 1.28\sqrt{(.16)(.84)/n};$$

$$n = \frac{[1.28\sqrt{(.14)(.86)} + 1.28\sqrt{(.16)(.84)}]^2}{(.14 - .16)^2} = 2086.$$

Substituting the value of $n = 2086$ into the first equation, we find $A = .1497$. The decision rule is: If $\bar{p} \leqslant .1497$, conclude $C_1$; if $\bar{p} > .1497$, conclude $C_2$.

(b) Since $\bar{p} = 228/2086 = .1093 \leqslant .1497$, conclude $C_1$.

lot, containing 1200 units ($n_2$), required 168 warranty adjustments. The manufacturer wishes to test whether or not the proportions of warranty adjustments under the two contracts are equal.

(a) Set up the alternative conclusions for this problem. $C_1$: _____ ; $C_2$: _____ √

(b) Assuming the two sets of data represent independent random samples, obtain the appropriate decision rule. The manufacturer wishes to control the risk of making a Type I error at .02. _____

_____

(c) What conclusion should be reached? _____

(d) If the two lots of product were produced under dissimilar conditions, in addition to having different warranty contracts, would this fact affect the conclusion drawn from the test? _____

_____

_____ √

## REVIEW PROBLEMS

1. A new safety program has been in effect for the current year just ended in each of the 20 production units of a company. Shown in Table 20.1 are the numbers of man-hours lost due to accidents in each of the units for the previous year ($X_1$), and for the current year under the new safety program ($X_2$).

**Table 20.1**

| Unit | $X_2$ | $X_1$ | Sign | Unit | $X_2$ | $X_1$ | Sign |
|---|---|---|---|---|---|---|---|
| #1 | 120 | 134 | – | #11 | 34 | 36 | – |
| #2 | 24 | 36 | – | #12 | 71 | 96 | – |
| #3 | 68 | 66 | + | #13 | 105 | 153 | – |
| #4 | 153 | 171 | – | #14 | 81 | 75 | + |
| #5 | 67 | 71 | – | #15 | 45 | 51 | – |
| #6 | 46 | 48 | – | #16 | 86 | 81 | ___ |
| #7 | 71 | 79 | – | #17 | 123 | 143 | ___ |
| #8 | 93 | 67 | + | #18 | 111 | 131 | ___ |
| #9 | 108 | 110 | – | #19 | 41 | 65 | ___ |
| #10 | 29 | 41 | – | #20 | 76 | 77 | ___ |

The company wishes to test whether or not the safety program has changed the number of man-hours lost due to accidents within the production units.

A simple test management may apply is called the *sign test*. In the sign test, a plus (+) sign is recorded if $X_2 - X_1 > 0$ and a minus sign (–) is recorded if $X_2 - X_1 < 0$, for each pair of observations in the sample. This

has been done for part of the data above. (If $X_1 = X_2$ for any pair, then that pair of observations is dropped from the analysis.) Let $\bar{p}$ be the proportion of plus signs in the sample. The alternative conclusions to be tested are: $C_1: p = 1/2, C_2: p \neq 1/2$. The conclusion $C_1$ is equivalent to stating that the population of differences $X_2 - X_1$ has a median of zero, which in turn implies that the new safety program on the average has not changed the number of man-hours lost in production units due to accidents.

(a) Complete the sign tabulation and calculate $\bar{p}$. _____ √

(b) Construct the appropriate decision rule assuming that the risk of a Type I error is to be controlled at .046.

_____

(c) What conclusion should be reached? _____ √

(d) Sketch the power curve for the decision rule constructed in (b). Use the power curve ordinates corresponding to $p = 0, .3, .5, .7$ and 1.0 in making your sketch. √

(e) What is implied about the population of differences $X_2 - X_1$ if $p = .7$? _____

_____

_____

If $p = .3$? _____ √

2. A producer and a consumer are negotiating the terms of a quality control sampling procedure for a large shipment of product. The alternative conclusions are: $C_1: p \leqslant .05$, $C_2: p > .05$, where $p$ is the proportion of defective items in the shipment. The consumer accepts the shipment if $C_1$ is concluded, and rejects the shipment if $C_2$ is concluded. It is agreed that the probability of concluding $C_2$ when $p = .05$ is to be .05 (this error specification frequently is called the *producer's risk*), and the probability of concluding $C_1$ when $p = .07$ is to be .05 (this error specification frequently is called the *consumer's risk*).

(a) State the two risk specifications symbolically. _____

_____

(b) To which type of error (I, II) is the producer's risk related? _____ The consumer's risk? _____ √

(c) What sample size and decision rule should be used in this decision problem in order to satisfy both risk specifications? $n =$ _____; decision rule: _____

_____ √

(d) For what value of $p$ is the probability of an acceptable shipment's being rejected a maximum? _____

(e) What is the maximum probability of an acceptable shipment's being rejected? _____

(f) What is the maximum producer's risk? _____ √

3. A researcher is studying preferences of consumers for two styles of furniture, A and B. In a random sample of 50 women ($n_1$), 36 prefer A. In an independent random sample of 50 men ($n_2$), 16 prefer A. The researcher desires to test whether or not the proportion preferring A is identical for men and women in the relevant populations. The risk of making a Type I error is to be held at .02.

(a) Set up the alternative conclusions for this problem. $C_1$: _____; $C_2$: _____ √

(b) What decision rule should the researcher use? _____

_____

(c) What conclusion should be reached? _____ √

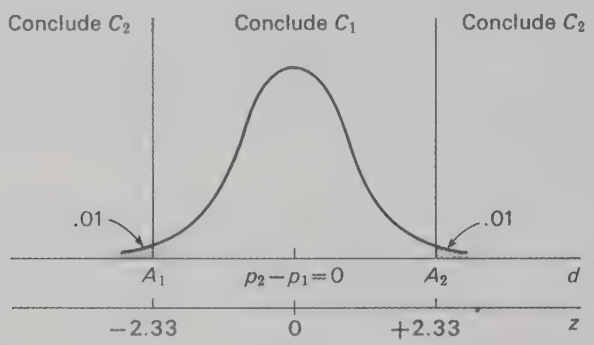
Answers

5. (a) $C_1: p_2 - p_1 = 0$; $C_2: p_2 - p_1 \neq 0$.

(b) $\bar{p}_1 = 545/5000 = .109$; $\bar{p}_2 = 168/1200 = .140$;

$\bar{p} = [5000(.109) + 1200(.140)]/(5000 + 1200) = .115$;

$s_d = \sqrt{(.115)(.885)/(5000 - 1) + (.115)(.885)/(1200 - 1)} = .0102$.




$A_1 = 0 - 2.33(.0102) = -.024$ and $A_2 = 0 + 2.33(.0102) = +.024$.

The decision rule is: If $-.024 \leq d \leq +.024$, conclude $C_1$; if $d < -.024$ or $d > +.024$, conclude $C_2$; or equivalently: If $-2.33 \leq (d - 0)/.0102 \leq +2.33$, conclude $C_1$; if $(d - 0)/.0102 < -2.33$ or $(d - 0)/.0102 > +2.33$, conclude $C_2$.

(c) Since $d = \bar{p}_2 - \bar{p}_1 = .140 - .109 = .031 > .024$, the manufacturer should conclude $C_2$.

(d) Yes, since the effect of different warranty contracts on the proportion of warranty adjustments would be mixed with the effect of different production conditions on the proportion of adjustments.

## ANSWERS TO REVIEW PROBLEMS

1. (a)

| Unit | Sign |
|---|---|
| #16 | + |
| #17 | − |
| #18 | − |
| #19 | − |
| #20 | − |

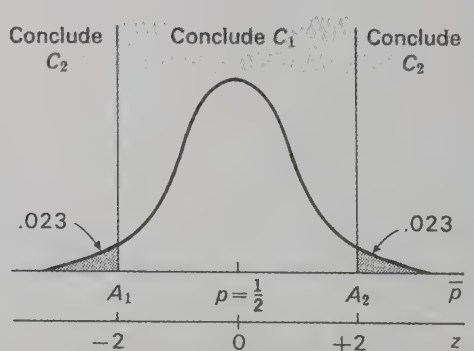
$\bar{p} = 4/20 = .20$.

(b) $p = 1/2$; $\sigma_{\bar{p}} = \sqrt{(1/2)(1/2)/20} = .112$.

$A_1 = 1/2 - 2(.112) = .276$;

$A_2 = 1/2 + 2(.112) = .724$.

The decision rule is: If $.276 \leq \bar{p} \leq .724$, conclude $C_1$; if $\bar{p} < .276$ or $\bar{p} > .724$, conclude $C_2$.



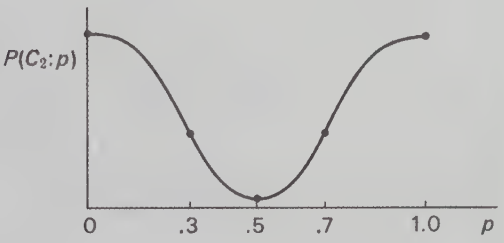
## ANSWERS TO REVIEW PROBLEMS

**1. cont'd.**

**(c)** Since $\bar{p} = .20 < .276$, conclude $C_2$.

**(d)**

| $p$ | $\sigma_{\bar{p}}$ | $z$ values | $P(C_2 : p)$ |
|---|---|---|---|
| 0 | 0 | — | 1.000 |
| .3 | .102 | − .23, +4.14 | .409 |
| .5 | .112 | −2.00, +2.00 | .046 |
| .7 | .102 | −4.14, + .23 | .409 |
| 1.0 | 0 | — | 1.000 |

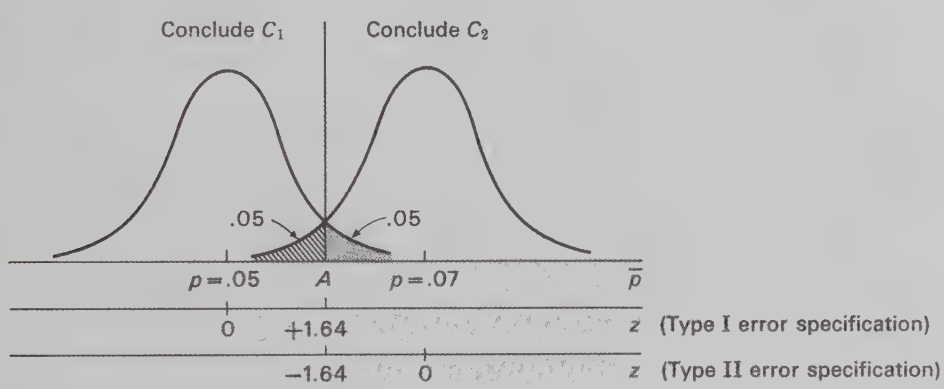



**(e)** $p = .7$ implies that 70 per cent of the $X_2 - X_1$ differences in the population are positive, hence $X_2$ is likely to be larger than $X_1$. If $p = .3$ the opposite is implied. Remember that a positive value for $X_2 - X_1$ indicates that the new safety program has increased the number of man-hours lost due to accidents in that unit.

**2. (a)** $P(C_2 : p = .05) = .05$ (producer's risk);
$P(C_1 : p = .07) = .05$ (consumer's risk).

**(b)** Producer's risk is related to a Type I error.
Consumer's risk is related to a Type II error.

**(c)**




$$A = .05 + 1.64\sqrt{(.05)(.95)/n};$$

$$A = .07 - 1.64\sqrt{(.07)(.93)/n}.$$

Solving these two equations for $A$ and $n$ yields:

$$n = \frac{[1.64\sqrt{(.05)(.95)} + 1.64\sqrt{(.07)(.93)}]^2}{(.05 - .07)^2} = 1504 \quad \text{and} \quad A = .0592.$$

The decision rule is: If $\bar{p} \leqslant .0592$, conclude $C_1$; if $\bar{p} > .0592$, conclude $C_2$.

**(d)** $p = .05$.

## ANSWERS TO REVIEW PROBLEMS

2. cont'd.

(e) The maximum probability of an acceptable shipment's being rejected is written symbolically as $P(C_2:\ p = .05)$. From (a), we see that $P(C_2: p = .05) = .05$, the Type I risk specification used in constructing the decision rule.

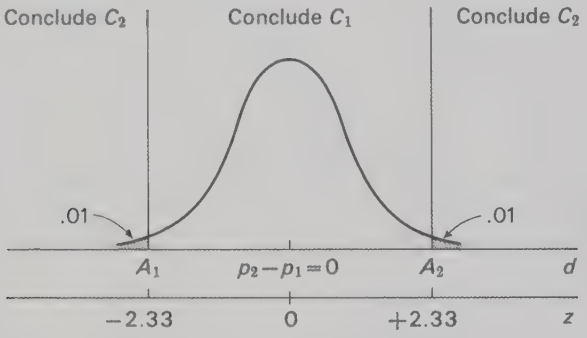
(f) $P(C_2: p = .05) = .05$ is the maximum producer's risk.

3. (a) $C_1: p_2 - p_1 = 0;\ \ C_2: p_2 - p_1 \neq 0.$

(b) $\bar{p}_1 = 36/50 = .72;\ \ \bar{p}_2 = 16/50 = .32;$

$\bar{p} = [50(.72) + 50(.32)]/(50 + 50) = .52;$

$s_d = \sqrt{(.52)(.48)/(50 - 1) + (.52)(.48)/(50 - 1)} = .1009.$




$A_1 = 0 - 2.33(.1009) = -.235$ and $A_2 = 0 + 2.33(.1009) = +.235.$

The decision rule is: If $-.235 \leq d \leq +.235$, conclude $C_1$; if $d < -.235$ or $d > +.235$, conclude $C_2$.

(c) Since $d = \bar{p}_2 - \bar{p}_1 = .32 - .72 = -.40 < -.235$, conclude $C_2$.

# 21

# Tests of Goodness of Fit (Chi-Square Tests)

In this section, we consider decision rules for testing whether or not the population sampled follows a particular probability distribution.

**A. Chi-square ($\chi^2$) Distribution.** The decision rules in this section require the $\chi^2$ (read: chi-square) probability distribution. The $\chi^2$ distribution has one parameter, called *degrees of freedom*, which we denote by $\delta$ (Greek delta). The mean and standard deviation of the $\chi^2$ distribution are $\mu = \delta$ and $\sigma = \sqrt{2\delta}$ respectively. There is a different $\chi^2$ distribution for each value of $\delta$. The $\chi^2$ distribution for four degrees of freedom ($\delta = 4$) is shown in Figure 21.1.

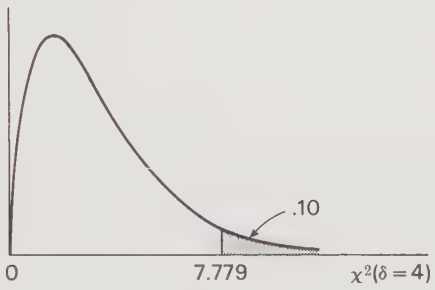



**Figure 21.1**

Since $\chi^2$ is a continuous random variable, probabilities relating to it correspond to areas under the curve. Table A-6 gives the $\chi^2$ values corresponding to specified areas in either the upper or lower tails, for various values of $\delta$.

Example 1 To illustrate the use of Table A-6, consider the $\chi^2$ value that corresponds to a right tail area of .10 for a distribution with four degrees of freedom. Consulting Table A-6 under $\delta = 4$ and a right tail area of .10, we see that the desired $\chi^2$ value equals 7.779. This value is shown in Figure 21.1.

**Problem 1**

Look up the following $\chi^2$ values in Table A-6:

(a) $\delta = 10$, right tail area = .05, $\chi^2 =$ _____ ;

(b) $\delta = 26$, right tail area = .01, $\chi^2 =$ _____ ;

(c) $\delta = 26$, left tail area = .01, $\chi^2 =$ _____ $\surd$

**B. Discrete Probability Distributions.** We illustrate tests concerning discrete probability distributions by considering the Poisson distribution.

Example 2 A researcher for a harbor control commission wishes to decide whether or not the number of ships that arrive in the harbor each hour follows a Poisson distribution with $\mu = 1.0$. The decision problem in this example

**Answers**

1. (a) 18.307.
   (b) 45.642.
   (c) 12.198.

therefore involves the following two alternatives:

$C_1$: The probability distribution is Poisson with $\mu = 1.0$.
$C_2$: The probability distribution is not Poisson with $\mu = 1.0$.

The researcher studied the number of ships arriving each hour for a random sample of 70 hours and obtained the sample frequencies, denoted by $f_s$, in Table 21.1. We see that no ships arrived in 42 of the sample hours, one ship in 21 of the sample hours, and so on.

**Table 21.1**

| Number of Ships Arriving Per Hour | $f_s$ | Probability | $f_e$ |
|---|---|---|---|
| 0 | 42 | .3679 | 25.753 |
| 1 | 21 | .3679 | 25.753 |
| 2 | 7 | .1839 | 12.873 |
| 3 or more | 0 | .0803 | 5.621 |
| Total | 70 | 1.0000 | 70.000 |

The probabilities for a Poisson distribution with $\mu = 1.0$ are obtained from Table A-2 and are shown in Table 21.1. The expected frequencies if $C_1$ holds are obtained by multiplying the Poisson probabilities for $\mu = 1.0$ by 70. The expected frequencies are denoted by $f_e$ in Table 21.1. For instance, for zero ships arriving per hour, the expected frequency is $f_e = 70 \times .3679 = 25.753$.

The test statistic used in the test is denoted $V_1$ and is:

$$V_1 = \Sigma \frac{(f_s - f_e)^2}{f_e}.$$

For the example considered here, we have:

**Table 21.2**

| Number of Ships Arriving Per Hour | Sample Frequency $f_s$ | Expected Frequency $f_e$ | Deviation $(f_s - f_e)$ | Squared Deviation $(f_s - f_e)^2$ | Relative Squared Deviation $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|
| 0 | 42 | 25.753 | 16.247 | 263.965 | 10.25 |
| 1 | 21 | 25.753 | − 4.753 | 22.591 | .88 |
| 2 | 7 | 12.873 | − 5.873 | 34.492 | 2.68 |
| 3 or more | 0 | 5.621 | − 5.621 | 31.596 | 5.63 |
| Total | 70 | 70.000 | 0 | | $V_1 = 19.44$ |

Thus, in the example, $V_1 = 19.44$.

To construct the decision rule so as to control the risk of a Type I error, we require the probability distribution of $V_1$ when $C_1$ is correct. Statistical theory indicates that if the sample size is large, specifically, if the expected frequency in each class is 5 or more, the probability distribution of $V_1$ is closely approximated by the $\chi^2$ distribution with $k - 1$ degrees of freedom when $C_1$ is correct. Here $k$ is the number of classes in the frequency distribution. In this example, $k$ is equal to 4, so if $C_1$ is correct, the distribution of $V_1$ is approximately $\chi^2$ with $4 - 1 = 3$ degrees of freedom.

Large values of $V_1$ lead to the conclusion $C_2$, since if the distribution is not the specified one (Poisson with $\mu = 1.0$ in our example), we expect the sample frequencies $f_s$ to diverge from the expected frequencies $f_e$, resulting in a large value of $V_1$.

Consequently, the appropriate type of decision rule for controlling the risk of a Type I error (level of significance) is the one-sided rule:

$$\text{If } V_1 \leqslant A, \text{ conclude } C_1; \quad \text{if } V_1 > A, \text{ conclude } C_2.$$

To obtain the action limit $A$ for our example, we employ the $\chi^2$ distribution as in Figure 21.2.

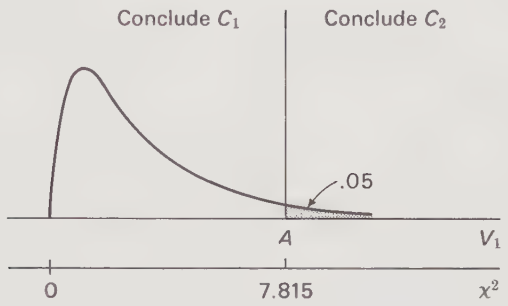



**Figure 21.2**

Here we are controlling the risk of a Type I error at .05. The $\chi^2$ value associated with a right tail area of .05 for three degrees of freedom is 7.815. Thus, the action limit of the decision rule is 7.815, and the decision rule is: If $V_1 \leqslant 7.815$, conclude $C_1$; if $V_1 > 7.815$, conclude $C_2$.

For our example, $V_1 = 19.44$, hence the decision rule leads us to conclude $C_2$, that the number of ships that arrive in the harbor each hour does not follow a Poisson distribution with $\mu = 1.0$.

**Problem 2**

Does the conclusion reached in Example 2 above imply that the distribution of ship arrivals is not Poisson? Explain. _____

_____

_____ √

**Problem 3**

The frequency distribution of the weekly demand for a product during the past 52 weeks is shown in Table 21.3. It is desired to test whether or not weekly demand follows a Poisson distribution with $\mu = 2.0$. The probability of a Type I error is to be controlled at .02.

**Answers**

2. No. The alternative $C_2$ includes two types of distributions: (1) Poisson distributions where the mean is not 1.0 and (2) distributions other than Poisson distributions.

**Table 21.3**

| Weekly Demand | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2 / f_e$ |
|---|---|---|---|---|---|
| 0 | 9 | 7.036 | 1.964 | 3.857 | .55 |
| 1 | 15 | 14.076 | .924 | .854 | .06 |
| 2 | 21 | 14.076 | 6.924 | 47.942 | 3.41 |
| 3 | 7 | ——— | ——— | ——— | ——— |
| 4 or more | 0 | ——— | ——— | ——— | ——— |
| Total | 52 | 52.000 | | | ——— |

(a) Specify the alternative conclusions for this decision problem. $C_1$: _____ ; $C_2$: _____ √

(b) Set up the appropriate decision rule. _____

(c) Determine the value of the test statistic by completing Table 21.3. What conclusion should be reached?

(d) Does your conclusion in (c) imply that the distribution of weekly demand is not Poisson? Explain. _____ √

Example 3 In Example 2, the researcher tested whether the probability distribution followed a particular functional form with a specified value for the parameter; specifically, a Poisson distribution with $\mu = 1.0$. At times, it is desired to test whether the probability distribution follows a particular functional form without specifying values for the parameters. For instance, in our example, the researcher may not have been concerned with the parameter of the distribution, but simply have wished to test whether the distribution is Poisson. In that event, the two alternatives would be:

$C_1$: The probability distribution is Poisson.
$C_2$: The probability distribution is not Poisson.

Since the parameter $\mu$ in the Poisson distribution has not been specified in $C_1$, it is necessary to estimate it from the observed sample data in order to obtain the expected frequencies $f_e$.

An appropriate estimator of $\mu$ is the sample mean $\bar{X}$. For the data in Example 2, the sample mean is calculated as follows:

**Table 21.4**

| Number of Ships Arriving Per Hour ($X$) | $f_s$ | $f_s X$ |
|---|---|---|
| 0 | 42 | 0 |
| 1 | 21 | 21 |
| 2 | 7 | 14 |
| 3 or more | 0 | 0 |
| Total | $n = 70$ | 35 |

$$\bar{X} = \frac{\Sigma f_s X}{n} = \frac{35}{70} = .5.$$

With $\overline{X} = .5$ used as an estimate of $\mu$, the expected frequencies are obtained from the probabilities for a Poisson distribution with a mean of .5. Both these Poisson probabilities and the expected frequencies are shown in Table 21.5.

**Table 21.5**

| Number of Ships Arriving Per Hour ($X$) | $f_s$ | Probability | $f_e$ |
|---|---|---|---|
| 0 | 42 | .6065 | 42.455 |
| 1 | 21 | .3033 | 21.231 |
| 2 | 7 | .0758 | 5.306 |
| 3 or more | 0 | .0144 | 1.008 |
| Total | 70 | 1.0000 | 70.000 |

At this point, we note that the last class has an expected frequency of 1.008, so that the requirement for the $\chi^2$ approximation of an expected frequency of 5 or more in each class is not met. We therefore need to combine classes so the expected frequency will be large enough. We combine the classes $X = 2$ and $X \geqslant 3$ into the single class $X \geqslant 2$, and the condition on expected frequencies is satisfied. For this combined class, $f_s = 7 + 0 = 7$ and $f_e = 5.306 + 1.008 = 6.314$. We now obtain $V_1$ as follows:

**Table 21.6**

| Number of Ships Arriving Per Hour ($X$) | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2 / f_e$ |
|---|---|---|---|---|---|
| 0 | 42 | 42.455 | −.455 | .207 | .005 |
| 1 | 21 | 21.231 | −.231 | .053 | .002 |
| 2 or more | 7 | 6.314 | +.686 | .471 | .075 |
| Total | 70 | 70.000 | 0 | | $V_1 = .082$ |

When $C_1$ holds, $V_1$ follows a $\chi^2$ distribution if the sample size is large, as noted earlier. However, for each parameter value of the probability distribution specified in $C_1$ that needs to be estimated from the sample, the degrees of freedom must be reduced by one. In general, if $m$ parameter values are estimated from the sample observations, then $V_1$ is distributed as a $\chi^2$ variable with $k - m - 1$ degrees of freedom when $C_1$ holds. In the present example, there are three classes; thus $k = 3$. The sample mean $\overline{X}$ was used to estimate the parameter $\mu$ of the Poisson distribution, hence $m = 1$. Thus, here, if $C_1$ holds, $V_1$ is distributed as a $\chi^2$ variable with $k - m - 1 = 3 - 1 - 1 = 1$ degree of freedom.

Assume the risk of a Type I error is to be held at .05. In Table A-6, we find the $\chi^2$ value associated with a right tail area of .05, for the $\chi^2$ distribution with one degree of freedom, to be 3.841. The decision rule therefore is: If $V_1 \leqslant 3.841$, conclude $C_1$; if $V_1 > 3.841$, conclude $C_2$. Since $V_1 = .082$, the researcher should conclude $C_1$, that the probability distribution is Poisson.

(In some books, a *correction for continuity* is applied to the formula for $V_1$ when there is one degree of freedom. This correction for continuity will not be used here.)

**Problem 4** (For answers, see page 178.)

Refer to Problem 3. Suppose the question of concern had simply been whether or not weekly demand follows a Poisson distribution.

(a) Specify the alternatives $C_1$ and $C_2$. $C_1$: _____;
$C_2$: _____ √

(b) What value of $\mu$ should be used for determining the expected frequencies? _____ Explain. _____
_____ √

(c) Calculate the expected frequencies. √

Answers

3. (a) $C_1$: The probability distribution is Poisson with $\mu = 2.0$;
$C_2$: The probability distribution is not Poisson with $\mu = 2.0$.

(b) Decision rule: If $V_1 \leq 11.668$, conclude $C_1$; if $V_1 > 11.668$, conclude $C_2$. Here 11.668 is the $\chi^2$ value for $k - 1 = 5 - 1 = 4$ degrees of freedom and a right tail area of .02.

(c)

| Weekly Demand | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|
| 3 | 7 | 9.381 | −2.381 | 5.669 | .60 |
| 4 or more | 0 | 7.431 | −7.431 | 55.220 | 7.43 |

$V_1 = .55 + \cdots + .60 + 7.43 = 12.05$.
Since $V_1 = 12.05 > 11.668$, conclude $C_2$.

(d) **No. The distribution of weekly demand may be a Poisson distribution with $\mu \neq 2.0$.**

(d) Is the requirement for the $\chi^2$ approximation met by the expected frequencies in (c)?____If not, what should be done? _____ √

(e) Calculate the appropriate test statistic. _____

(f) How many degrees of freedom are there for the relevant $\chi^2$ distribution?____ √

(g) Set up the appropriate decision rule. Assume the probability of a Type I error is to be controlled at .02.

(h) What conclusion should be reached? _____ √

**C. Continuous Probability Distributions.** We illustrate tests concerning continuous probability distributions by considering the normal distribution.

Example 4 A mail order firm wishes to test whether the number of orders received each eight-hour day follows a normal distribution. The alternative conclusions here are: $C_1$: The probability distribution is normal; $C_2$: The probability distribution is not normal.

Data on the number of orders received each day are available for fifty days, and are shown in Table 21.7. Since the normal distribution has two parameters, $\mu$ and $\sigma$, and since these are not specified in $C_1$, the sample mean $\bar{X}$ and sample standard deviation $s$ are used instead. These sample estimates are $\bar{X} = 750$ orders and $s = 250$ orders (calculations not shown).

To illustrate the test procedure, we employ five classes, each containing the same expected frequency. The class limits are obtained by dividing the normal distribution, with a mean of 750 and a standard deviation of 250, into five regions where each region spans 20% of the area under the normal curve. Thus, we need the $z$ values corresponding to the 20th, 40th, 60th, and 80th percentiles. These are indicated on the $z$ scale in Figure 21.3.

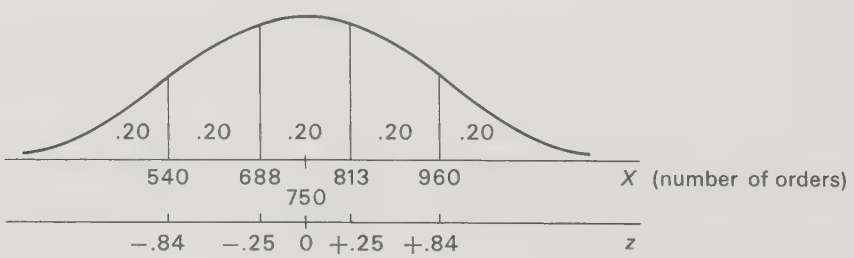



**Figure 21.3**

For instance, a $z$ value of $-.84$ corresponds to a lower tail area of .20 in the normal distribution. Since the mean and standard deviation of the normal distribution are 750 and 250 respectively, the limit equals $750 - .84(250) = 540$. The other class limits are calculated in a similar manner, and are shown in the diagram. The resulting five classes are shown in Table 21.7.

**Table 21.7**

| Number of Orders $X$ | $f_s$ | $f_e$ |
|---|---|---|
| Under 540 | 8 | 10 |
| 540–under 688 | 12 | 10 |
| 688–under 813 | 15 | 10 |
| 813–under 960 | 8 | 10 |
| 960 and over | 7 | 10 |
| Total | 50 | 50 |

Since each class corresponds to 20% of the area under the normal curve, 20% of the 50 sample observations are expected to fall in each class if $C_1$ holds, that is, $f_e = 10$ for each class.

The sample frequencies for the five classes are shown under the caption $f_s$. (We already stated that the mean and standard deviation of these 50 sample observations are 750 and 250 respectively.) From this point on, the test proceeds as in Example 1 and Example 2. See if you can complete the test by working Problem 5.

**Problem 5**

(a) Calculate the test statistic $V_1$. _____

(b) How many degrees of freedom are associated with the relevant $\chi^2$ distribution? _____ √

(c) Set up the appropriate decision rule. Assume the probability of a Type I error is to be held at .05.

_____

(d) What conclusion should be drawn? _____ √

**Problem 6** (For answers, see page 180.)

The lifetimes in hours of a sample of forty-watt incandescent lamps in quality control forced-life tests were as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1105 | 1243 | 1204 | 1203 | 1310 | 1262 | 1234 | 1104 |
| 1303 | 1185 | 759 | 1404 | 944 | 1343 | 932 | 1055 |
| 1381 | 816 | 1067 | 1252 | 1248 | 1324 | 1000 | 984 |

For the above sample data, $\bar{X} = 1153$ hours, $s = 176$ hours, and $n = 24$.

It is desired to test whether the lifetimes of the lamps are normally distributed, and the probability of a Type I error is to be held at .10.

(a) Specify the alternative conclusions. $C_1$: _____;
$C_2$: _____ √

(b) Four classes with equal expected frequencies are to be employed in the test. What are the $z$ values required for finding the class limits? _____ _____ _____ √

(c) Set up the appropriate classes and determine $f_s$ and $f_e$. √

(Problem 6 continued on bottom of page 178.)

Answers

4. (a) $C_1$: The probability distribution is Poisson;
$C_2$: The probability distribution is not Poisson.

(b) The sample mean $\bar{X}$ is used to estimate the population mean $\mu$ because $\mu$ is not specified in $C_1$. Here $\bar{X} = [9(0) + 15(1) + 21(2) + 7(3)]/52 = 1.5$.

(c)

| Weekly Demand ($X$) | $f_s$ | Probability | $f_e$ |
|---|---|---|---|
| 0 | 9 | .2231 | 11.601 |
| 1 | 15 | .3347 | 17.404 |
| 2 | 21 | .2510 | 13.052 |
| 3 | 7 | .1255 | 6.526 |
| 4 or more | 0 | .0657 | 3.417 |
| Total | 52 | 1.0000 | 52.000 |

(d) No. The class "$X \geqslant 4$" has an expected frequency less than five. Thus, we combine the classes "$X = 3$" and "$X \geqslant 4$" into one class, "$X \geqslant 3$".

(e)

| Weekly Demand | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|
| 0 | 9 | 11.601 | −2.601 | 6.765 | .58 |
| 1 | 15 | 17.404 | −2.404 | 5.779 | .33 |
| 2 | 21 | 13.052 | 7.948 | 63.171 | 4.84 |
| 3 or more | 7 | 9.943 | −2.943 | 8.661 | .87 |
| Total | 52 | 52.000 | 0 | | $V_1 = 6.62$ |

(f) $k - m - 1 = 4 - 1 - 1 = 2$ degrees of freedom.

(g) The $\chi^2$ value for 2 degrees of freedom and a right tail area of .02 is 7.824. The decision rule is: If $V_1 \leqslant 7.824$, conclude $C_1$; if $V_1 > 7.824$, conclude $C_2$.

(h) Since $V_1 = 6.62 \leqslant 7.824$, conclude $C_1$.

5. (a)

| Number of Orders | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|
| Under 640 | 8 | 10 | −2 | 4 | .4 |
| 640–under 688 | 12 | 10 | 2 | 4 | .4 |
| 688–under 813 | 15 | 10 | 5 | 25 | 2.5 |
| 813–under 960 | 8 | 10 | −2 | 4 | .4 |
| 960 and over | 7 | 10 | −3 | 9 | .9 |
| Total | 50 | 50 | 0 | | $V_1 = 4.6$ |

(b) $k - m - 1 = 5 - 2 - 1 = 2$ degrees of freedom.

(c) The $\chi^2$ value for 2 degrees of freedom and a right tail area of .05 is 5.991. The decision rule is: If $V_1 \leqslant 5.991$, conclude $C_1$; if $V_1 > 5.991$, conclude $C_2$.

(d) Since $V_1 = 4.6 \leqslant 5.991$, conclude $C_1$.

(d) Calculate the appropriate test statistic. _____

(e) Set up the decision rule. _____

_____

(f) What conclusion should be reached? _____ √

## REVIEW PROBLEMS

1. It is desired to test whether the number of machines arriving at a repair facility each day follows a Poisson distribution. In a simple random sample of 100 days, the following frequency distribution was observed:

**Table 21.8**

| Number of Machines Arriving for Repairs Each Day | Frequency |
|---|---|
| 0 | 41 |
| 1 | 34 |
| 2 | 20 |
| 3 | 4 |
| 4 | 1 |
| Total | 100 |

(a) Specify the alternative conclusions. $C_1$: _____ ;
$C_2$: _____ √

(b) Calculate the test statistic. _____

(c) Set up the appropriate decision rule to control the Type I error probability at .10. _____
_____

(d) What conclusion should be reached? _____ √

2. In statistical analysis, it frequently is assumed that the population under study is normal. Consider the following sample observations obtained from a statistical study:

| | | | | |
|---|---|---|---|---|
| 65 | 51 | 15 | 36 | 39 |
| 37 | 18 | 45 | 49 | 47 |
| 17 | 46 | 37 | 36 | 32 |
| 34 | 36 | 40 | 40 | 65 |

$\bar{X} = 39.25$ $s = 13.29$

It is desired to test whether or not the above observations come from a normal population. The probability of making a Type I error is to be controlled at .05. (For answers, see page 182.)

(a) Specify the alternative conclusions. $C_1$: _____ ;
$C_2$: _____ √

(b) How many classes, containing equal expected frequencies, should be used? _____ Explain. _____
_____
_____ √

(c) Conduct the test.

$V_1$ = _____ ; decision rule: _____
_____ √

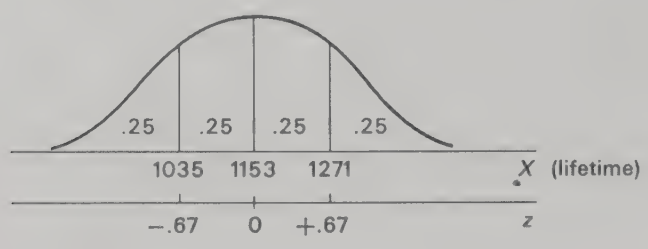
(Review Problem 2 continued on page 181.)

Answers

6. (a) $C_1$: The probability distribution is normal;
$C_2$: The probability distribution is not normal.

(b) $-.67, 0, +.67$.

(c)




(d)

| Lifetime | Tally | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|---|
| Under 1035 | //// / | 6 | 6 | 0 | 0 | 0 |
| 1035–under 1153 | //// | 4 | 6 | −2 | 4 | .67 |
| 1153–under 1271 | //// /// | 8 | 6 | +2 | 4 | .67 |
| 1271 and over | //// / | 6 | 6 | 0 | 0 | 0 |
| Total | | 24 | 24 | 0 | | $V_1 = 1.33$ |

(e) The decision rule is: If $V_1 \leq 2.706$, conclude $C_1$; if $V_1 > 2.706$, conclude $C_2$. Here 2.706 is the $\chi^2$ value for $k - m - 1 = 4 - 2 - 1 = 1$ degree of freedom and a right tail area of .10.

(f) Since $V_1 = 1.33 \leq 2.706$, conclude $C_1$.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $C_1$: The probability distribution is Poisson;
$C_2$: The probability distribution is not Poisson.

(b) **The estimate of the population mean $\mu$ is $[41(0) + 34(1) + \cdots + 1(4)]/100 = 90/100 = .9$. The probabilities in the following table are for a Poisson distribution with a mean of .9.**

| Number of Machines | $f_s$ | Probability | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|---|
| 0 | 41 | .4066 | 40.66 | .34 | .1156 | .003 |
| 1 | 34 | .3659 | 36.59 | −2.59 | 6.7081 | .183 |
| 2 | 20 | .1647 | 16.47 | 3.53 | 12.4609 | .757 |
| 3 or more | 5 | .0628 | 6.28 | −1.28 | 1.6384 | .261 |
| Total | 100 | 1.0000 | 100.00 | 0 | | $V_1 = 1.204$ |

**Note that the classes "3" and "4 or more" have been combined so that all expected frequencies are 5 or more.**

(c) The decision rule is: If $V_1 \leq 4.605$, conclude $C_1$; if $V_1 > 4.605$, conclude $C_2$. Here 4.605 is the $\chi^2$ value for a right tail area of .10 and $k - m - 1 = 4 - 1 - 1 = 2$ degrees of freedom.

(d) Since $V_1 = 1.204 \leq 4.605$, conclude $C_1$.

(d) With the decision rule constructed in (c), is there a higher probability of concluding $C_2$ when the actual probability distribution is approximately (but not exactly) normal or when the actual probability distribution is highly skewed? Explain. _____

_____

_____

(e) What generalization is suggested by your answer to (d)? _____

_____

_____

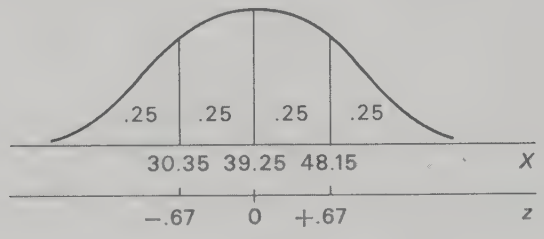
_____ √

## ANSWERS TO REVIEW PROBLEMS

2. (a) $C_1$: The probability distribution is normal;
   $C_2$: The probability distribution is not normal.

   (b) **Four. In this problem, the largest number of classes with equal expected frequencies of at least five is four classes. Fewer classes would not permit a test since no degrees of freedom would be available.**

   (c) The estimates of the population mean $\mu$ and standard deviation $\sigma$ are $\overline{X} = 39.25$ and $s = 13.29$ respectively.




Here $39.25 - .67(13.29) = 30.35$ and $39.25 + .67(13.29) = 48.15$. The four classes, with the corresponding sample and expected frequencies, are shown below:

| Class | Tally | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|---|
| Under 30.35 | /// | 3 | 5 | −2 | 4 | .8 |
| 30.35–under 39.25 | ~~////~~ /// | 8 | 5 | 3 | 9 | 1.8 |
| 39.25–under 48.15 | ~~////~~ | 5 | 5 | 0 | 0 | 0 |
| 48.15 and over | //// | 4 | 5 | −1 | 1 | .2 |
| Total | | 20 | 20 | 0 | | $V_1 = 2.8$ |

The decision rule is: If $V_1 \leq 3.841$, conclude $C_1$; if $V_1 > 3.841$, conclude $C_2$. Here 3.841 is the $\chi^2$ value for a right tail area of .05 and $k - m - 1 = 4 - 2 - 1 = 1$ degree of freedom. Since $V_1 = 2.8 \leq 3.841$, conclude $C_1$.

   (d) There is a higher probability of concluding $C_2$ when the actual probability distribution is highly skewed since $f_s$ and $f_e$ will tend to be further apart so that $V_1$ will tend to be larger.

   (e) The closer an alternative distribution is to the normal distribution specified in $C_1$, the lower the power of the decision rule to detect the departure from normality. The more an alternative distribution differs from the one specified in $C_1$, the higher the power of the decision rule to detect the departure.

# 22

# Contingency Tables (Chi-Square Tests)

In this section, we consider additional tests which employ the $\chi^2$ distribution and the test statistic $V_1$. (Readers who have not yet covered Section 21 should read, in that section, Topic A and Example 2 in Topic B.)

**A. Test of Statistical Independence.** First, we consider a statistical test as to whether the two variables in a bivariate classification are statistically independent.

Example 1 A food processing firm selected a random sample of 200 households, and classified them by income and whether they usually purchase cake mix or mix their own ingredients. The bivariate classification of the sample data shown in Table 22.1 is frequently called a *contingency table*.

**Table 22.1**

| | | | Family Income ($) | | | |
|---|---|---|---|---|---|---|
| | | | $B_1$: under 5000 | $B_2$: 5000–10,000 | $B_3$: over 10,000 | Total |
| Cake Mix Status | $A_1$: | Purchase | 24 | 100 | 36 | 160 |
| | $A_2$: | Mix Own | 16 | 20 | 4 | 40 |
| | | Total | 40 | 120 | 40 | 200 |

The firm wishes to test whether or not family income ($B$ variable) and cake mix status ($A$ variable) are statistically independent variables. The two alternative conclusions here are:

$C_1$: Family income and cake mix status are statistically independent.
$C_2$: Family income and cake mix status are not statistically independent.

If $C_1$ holds–if the two variables are statistically independent–the probability of any joint outcome in the table is equal to the product of the corresponding marginal probabilities. Thus, for example, $P(A_1 \text{ and } B_1) = P(A_1)\,P(B_1)$ with statistical independence. We estimate the marginal probabilities directly from the data in Table 22.1. For example: $\bar{P}(A_1) = 160/200 = .80$, $\bar{P}(B_1) = 40/200 = .20$, and so on. ($\bar{P}$ denotes an estimated probability.) We then estimate the joint probabilities, if independence holds; for example: $\bar{P}(A_1 \text{ and } B_1) = \bar{P}(A_1)\,\bar{P}(B_1) = (.80)\,(.20) = .16$. Finally, we estimate the expected frequency, if $C_1$ holds, by multiplying each estimated joint probability by 200. For instance, if $C_1$ holds, the estimated expected frequency for the joint outcome, $A_1$ and $B_1$, is $(.16)\,(200) = 32$.

**Problem 1**

For Example 1, find the estimated joint probabilities, assuming the two variables are independent, and find the estimated expected frequencies. Set these values up in Table 22.2. (For answers, see page 186.)

**Table 22.2**

| | Estimated Joint Probabilities | | | | | Estimated Expected Frequencies | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | Total | | $B_1$ | $B_2$ | $B_3$ | Total |
| $A_1$ | .16 | ___ | ___ | ___ | $A_1$ | 32 | ___ | ___ | ___ |
| $A_2$ | ___ | ___ | ___ | ___ | $A_2$ | ___ | ___ | ___ | ___ |
| Total | | | | | Total | | | | |

Example 2 We continue with Example 1. The sample frequencies $f_s$ and the estimated expected frequencies $f_e$ are displayed in Table 22.3.

**Table 22.3**

| Joint Classification | $f_s$ | Joint Probability | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|---|
| $A_1$ and $B_1$ | 24 | .16 | 32 | −8 | 64 | 2.00 |
| $A_1$ and $B_2$ | 100 | .48 | 96 | 4 | 16 | .17 |
| $A_1$ and $B_3$ | 36 | .16 | 32 | 4 | 16 | .50 |
| $A_2$ and $B_1$ | 16 | .04 | 8 | 8 | 64 | 8.00 |
| $A_2$ and $B_2$ | 20 | .12 | 24 | −4 | 16 | .67 |
| $A_2$ and $B_3$ | 4 | .04 | 8 | −4 | 16 | 2.00 |
| Total | 200 | 1.00 | 200 | 0 | | $V_1 = 13.34$ |

The test statistic $V_1$ is defined as in goodness of fit tests. For our example, it is $V_1 = 13.34$. If $C_1$ holds and if the expected frequencies are all 5 or more, the distribution of $V_1$ is approximately a $\chi^2$ distribution with $(R-1)(C-1)$ degrees of freedom. Here $R$ is the number of classes into which the row variable is classified and $C$ is the number of classes of the column variable.

Again, large values of $V_1$ lead to conclusion $C_2$. Suppose the firm wishes to control the probability of a Type I error (level of significance) at .05. Then, the appropriate decision rule for this example is: If $V_1 \leqslant 5.991$, conclude $C_1$; if $V_1 > 5.991$, conclude $C_2$. Here 5.991 is the $\chi^2$ value for $(R-1)(C-1) = (2-1)(3-1) = 2$ degrees of freedom and a right tail area of .05.

Since $V_1 = 13.34 > 5.991$, conclude $C_2$, that family income and cake-mix status are not statistically independent.

**Problem 2**

A clothing stylist is studying the reactions of shoppers to three new styles in men's clothing, $A$, $B$, and $C$. A random sample of 300 shoppers was interviewed and asked to indicate which of the three styles is preferred. The results are shown in Table 22.4, with shoppers classified according to: woman shopping alone, man shopping alone, and man and woman shopping together.

**Table 22.4**

| | | Shopper's Status | | | |
|---|---|---|---|---|---|
| | | Woman Alone | Man Alone | Man & Woman | Total |
| Preferred Style | $A$ | 4 | 30 | 26 | 60 |
| | $B$ | 35 | 115 | 60 | 210 |
| | $C$ | 21 | 5 | 4 | 30 |
| | Total | 60 | 150 | 90 | 300 |

It is desired to test whether or not shopper's status and style preference are statistically independent variables, with the probability of a Type I error controlled at .02.

(a) State the alternative conclusions. $C_1$: _____ ; $C_2$: _____ √

(b) Calculate the estimated joint probabilities and expected frequencies, if $C_1$ holds. √

(c) Calculate the appropriate test statistic. _____

(d) How many degrees of freedom are there for the relevant $\chi^2$ distribution? _____ √

(e) Set up the decision rule. _____

(f) What conclusion should be reached? _____ √

**B. Test of Homogeneity of Populations.** Here we shall test whether or not several populations have identical probabilities.

Example 3 Simple random samples of 200 items are selected independently from each of three large shipments. The number of defective items in each sample is determined. The sample results are shown in Table 22.5.

**Table 22.5**

| | Shipment | | | |
|---|---|---|---|---|
| | #1 | #2 | #3 | Total |
| Acceptable ($A$) | 190 | 180 | 194 | 564 |
| Defective ($D$) | 10 | 20 | 6 | 36 |
| Total | 200 | 200 | 200 | 600 |

It is desired to test whether the proportions of defective and acceptable items are the same in all three shipments. The alternative conclusions are:

$C_1$: Population proportions of defective and acceptable items are the same in all three shipments.
$C_2$: Population proportions of defective and acceptable items are not the same in all three shipments.

To obtain estimated expected frequencies when $C_1$ holds, we pool the three samples and estimate the proportions directly from the marginal totals in Table 22.5 as follows:

Estimated proportion acceptable = $\bar{p}_A$ = 564/600 = .94.
Estimated proportion defective = $\bar{p}_D$ = 36/600 = .06.

Here $\bar{p}$ denotes the estimated proportion. The estimated expected frequencies are obtained by multiplying these estimated proportions by the sample size for each shipment, 200 in this case. For instance, the estimated expected frequency of defective items in the sample from shipment #1 is 200(.06) = 12. The estimated expected frequencies are shown in Table 22.6. Given the sample frequencies and the estimated expected frequencies, the test statistic $V_1$ is calculated in the usual manner, as shown in Table 22.6.

Answers

1.

| | Estimated Joint Probabilities | | | | | Estimated Expected Frequencies | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $B_1$ | $B_2$ | $B_3$ | Total | | $B_1$ | $B_2$ | $B_3$ | Total |
| $A_1$ | .16 | .48 | .16 | .80 | $A_1$ | 32 | 96 | 32 | 160 |
| $A_2$ | .04 | .12 | .04 | .20 | $A_2$ | 8 | 24 | 8 | 40 |
| Total | .20 | .60 | .20 | 1.00 | Total | 40 | 120 | 40 | 200 |

2. (a) $C_1$: Shopper's status and style are statistically independent;
$C_2$: Shopper's status and style are not statistically independent.

(b) **The estimated joint probability distribution, if independence holds, is:**

| | | Shopper's Status | | | |
|---|---|---|---|---|---|
| | | Woman Alone | Man Alone | Man & Woman | Total |
| Preferred Style | $A$ | (.2)(.2) = .04 | .10 | .06 | .20 |
| | $B$ | .14 | .35 | .21 | .70 |
| | $C$ | .02 | .05 | .03 | .10 |
| | Total | .20 | .50 | .30 | 1.00 |

**The estimated expected frequencies are obtained by multiplying the above joint probabilities by 300.**

(c)

| Joint Classification | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|
| $A$ & woman | 4 | 12 | − 8 | 64 | 5.33 |
| $B$ & woman | 35 | 42 | − 7 | 49 | 1.17 |
| $C$ & woman | 21 | 6 | 15 | 225 | 37.50 |
| $A$ & man | 30 | 30 | 0 | 0 | 0 |
| $B$ & man | 115 | 105 | 10 | 100 | .95 |
| $C$ & man | 5 | 15 | −10 | 100 | 6.67 |
| $A$ & (man & woman) | 26 | 18 | 8 | 64 | 3.56 |
| $B$ & (man & woman) | 60 | 63 | − 3 | 9 | .14 |
| $C$ & (man & woman) | 4 | 9 | − 5 | 25 | 2.78 |
| Total | 300 | 300 | 0 | | $V_1 = 58.10$ |

(d) $(R - 1)(C - 1) = (3 - 1)(3 - 1) = 4$ degrees of freedom.

(e) The decision rule is: If $V_1 \leqslant 11.668$, conclude $C_1$; if $V_1 > 11.668$, conclude $C_2$.

(f) Since $V_1 = 58.10 > 11.668$, conclude $C_2$.

**Table 22.6**

| Joint Classification | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|
| Acceptable & Shipment #1 | 190 | 188 | 2 | 4 | .02 |
| Acceptable & Shipment #2 | 180 | 188 | −8 | 64 | .34 |
| Acceptable & Shipment #3 | 194 | 188 | 6 | 36 | .19 |
| Defective & Shipment #1 | 10 | 12 | −2 | 4 | .33 |
| Defective & Shipment #2 | 20 | 12 | 8 | 64 | 5.33 |
| Defective & Shipment #3 | 6 | 12 | −6 | 36 | 3.00 |
| Total | 600 | 600 | 0 | | $V_1 = 9.21$ |

If $C_1$ holds and the expected frequencies are all 5 or more, the distribution of $V_1$ is approximately a $\chi^2$ distribution with $(k-1)(m-1)$ degrees of freedom. Here $k$ is the number of classes (the same for all three populations) and $m$ is the number of populations. In this example, $k = 2$, and $m = 3$. Thus, $V_1$ has an approximate $\chi^2$ distribution with $(2-1)(3-1) = 2$ degrees of freedom when $C_1$ holds.

As usual, large values of $V_1$ lead to conclusion $C_2$. Suppose the probability of a Type I error is to be controlled at .10. Then, the appropriate decision rule for this example is: If $V_1 \leqslant 4.605$, conclude $C_1$; if $V_1 > 4.605$, conclude $C_2$. Here 4.605 is the $\chi^2$ value for 2 degrees of freedom and a right tail area of .10.

Since $V_1 = 9.21 > 4.605$, we conclude $C_2$, that the proportions of defective and acceptable items are not the same in all three shipments.

**Problem 3**

A company has prepared a questionnaire to measure how well persons are informed about the company. Scores on the questionnaire can range from 0 to 100; a high score indicates considerable knowledge about the company and its product line. Random samples of 100 persons from each of three income groups were interviewed. The results are given in Table 22.7.

**Table 22.7**

| | | Income ($) | | | |
|---|---|---|---|---|---|
| | | Under 5000 | 5000–9999 | 10,000 and over | Total |
| Score | 0–29 | 70 | 60 | 20 | 150 |
| | 30–69 | 20 | 30 | 40 | 90 |
| | 70–100 | 10 | 10 | 40 | 60 |
| | Total | 100 | 100 | 100 | 300 |

The company wishes to determine whether the proportions of persons falling into the three score classes are the same for all income groups. It is desired to control the probability of a Type I error at .05.

(a) State the alternative conclusions.
$C_1$: _____;
$C_2$: _____ √

(b) Calculate the estimated proportions and expected frequencies if $C_1$ holds. √

(c) Calculate the test statistic. _____

(d) How many degrees of freedom are there for the relevant $\chi^2$ distribution? _____ √

(e) Set up the appropriate decision rule. _____
_____

(f) What conclusion should be reached? _____

(g) Which income group is best informed about the company? _____ √

## REVIEW PROBLEMS

1. An oil company wishes to test whether or not the volume of customer traffic at service stations is statistically independent of the corner of the road intersection on which the service station is located. The cross-classified data in Table 22.8 were obtained from a study of a random sample of 400 service stations located at road intersections.

**Table 22.8**

| | | Traffic Volume | | | |
|---|---|---|---|---|---|
| | | Heavy | Medium | Light | Total |
| Corner | NW | 30 | 40 | 50 | 120 |
| | NE | 15 | 30 | 55 | 100 |
| | SW | 10 | 20 | 50 | 80 |
| | SE | 25 | 30 | 45 | 100 |
| | Total | 80 | 120 | 200 | 400 |

Perform the test of statistical independence. Control the Type I error probability at .10. What conclusion should be drawn? √

2. A soft drink company is studying the reaction of teenagers to two different advertising schemes. The reaction of the respondent is rated as favorable, unfavorable, or no opinion. Two independent random samples of 50 teenagers are employed, with each scheme shown to one group. The reactions of the teenagers are summarized in Table 22.9.

**Table 22.9**

| | | Reaction | | | |
|---|---|---|---|---|---|
| | | Favorable | Unfavorable | No Opinion | Total |
| Scheme | *A* | 23 | 7 | 20 | 50 |
| | *B* | 27 | 9 | 14 | 50 |
| | Total | 50 | 16 | 34 | 100 |

Test whether or not the population proportions of teenagers in the reaction categories are the same for both advertising schemes. Control the Type I error probability at .05. √

Answers

3. (a) $C_1$: The population proportions of persons in the three score classes are the same for all income groups.

$C_2$: The population proportions of persons in the three score classes are not the same for all income groups.

(b) If $C_1$ holds, the estimated proportions in the three score classes are:

| Score Class | Estimated Proportion |
|---|---|
| 0 – 29 | 150/300 = .5 |
| 30 – 69 | 90/300 = .3 |
| 70 – 100 | 60/300 = .2 |
| Total | 1.0 |

The estimated expected frequencies are:

| | | Income ($) | | | |
|---|---|---|---|---|---|
| | | Under 5000 | 5000–9999 | 10,000 and Over | Total |
| Score | 0 – 29 | 50 | 50 | 50 | 150 |
| | 30 – 69 | 30 | 30 | 30 | 90 |
| | 70 – 100 | 20 | 20 | 20 | 60 |
| | Total | 100 | 100 | 100 | 300 |

(c)

| Joint Classification | | | | | | |
|---|---|---|---|---|---|---|
| Score Class | Income Group | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
| 0 – 29 | Under 5000 | 70 | 50 | 20 | 400 | 8.00 |
| 30 – 69 | Under 5000 | 20 | 30 | –10 | 100 | 3.33 |
| 70 – 100 | Under 5000 | 10 | 20 | –10 | 100 | 5.00 |
| 0 – 29 | 5000–9999 | 60 | 50 | 10 | 100 | 2.00 |
| 30 – 69 | 5000–9999 | 30 | 30 | 0 | 0 | 0 |
| 70 – 100 | 5000–9999 | 10 | 20 | –10 | 100 | 5.00 |
| 0 – 29 | 10,000 and over | 20 | 50 | –30 | 900 | 18.00 |
| 30 – 69 | 10,000 and over | 40 | 30 | 10 | 100 | 3.33 |
| 70 – 100 | 10,000 and over | 40 | 20 | 20 | 400 | 20.00 |
| Total | | 300 | 300 | 0 | | $V_1 = 64.67$ |

(d) $(k - 1)(m - 1) = (3 - 1)(3 - 1) = 4$ degrees of freedom.

(e) The decision rule is: If $V_1 \leqslant 9.488$, conclude $C_1$; if $V_1 > 9.488$, conclude $C_2$.

(f) Since $V_1 = 64.67 > 9.488$, conclude $C_2$.

(g) Having concluded $C_2$, an examination of the frequency distributions indicates that persons in the highest income group have the highest scores and therefore this group is best informed.

## ANSWERS TO REVIEW PROBLEMS

**1.** $C_1$: Customer traffic volume and corner location are statistically independent;
$C_2$: Customer traffic volume and corner location are not statistically independent.

The sample frequencies, estimated expected frequencies, and calculation of $V_1$ are shown below:

| Joint Classification | | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|---|
| Corner | Traffic | | | | | |
| NW | Heavy | 30 | 24 | 6 | 36 | 1.50 |
| NE | Heavy | 15 | 20 | − 5 | 25 | 1.25 |
| SW | Heavy | 10 | 16 | − 6 | 36 | 2.25 |
| SE | Heavy | 25 | 20 | 5 | 25 | 1.25 |
| NW | Medium | 40 | 36 | 4 | 16 | .44 |
| NE | Medium | 30 | 30 | 0 | 0 | 0 |
| SW | Medium | 20 | 24 | − 4 | 16 | .67 |
| SE | Medium | 30 | 30 | 0 | 0 | 0 |
| NW | Light | 50 | 60 | −10 | 100 | 1.67 |
| NE | Light | 55 | 50 | 5 | 25 | .50 |
| SW | Light | 50 | 40 | 10 | 100 | 2.50 |
| SE | Light | 45 | 50 | − 5 | 25 | .50 |
| Total | | 400 | 400 | 0 | | $V_1 = 12.53$ |

The decision rule is: If $V_1 \leqslant 10.645$, conclude $C_1$; if $V_1 > 10.645$, conclude $C_2$. Here 10.645 is the $\chi^2$ value for $(R-1)(C-1) = (4-1)(3-1) = 6$ degrees of freedom and a right tail area of .10. Since $V_1 = 12.53 > 10.645$, conclude $C_2$.

**2.** $C_1$: The population proportions of teenagers in the reaction classes are the same for both advertising schemes;

$C_2$: The population proportions are not the same for both advertising schemes.

The sample frequencies, estimated expected frequencies, and calculation of $V_1$ are shown below:

| Joint Classification | | $f_s$ | $f_e$ | $(f_s - f_e)$ | $(f_s - f_e)^2$ | $(f_s - f_e)^2/f_e$ |
|---|---|---|---|---|---|---|
| Scheme | Reaction | | | | | |
| *A* | Favorable | 23 | 25 | −2 | 4 | .16 |
| *B* | Favorable | 27 | 25 | 2 | 4 | .16 |
| *A* | Unfavorable | 7 | 8 | −1 | 1 | .12 |
| *B* | Unfavorable | 9 | 8 | 1 | 1 | .12 |
| *A* | No opinion | 20 | 17 | 3 | 9 | .53 |
| *B* | No opinion | 14 | 17 | −3 | 9 | .53 |
| Total | | 100 | 100 | 0 | | $V_1 = 1.63$ |

The decision rule is: If $V_1 \leqslant 5.991$, conclude $C_1$; if $V_1 > 5.991$, conclude $C_2$. Here 5.991 is the $\chi^2$ value for $(k-1)(m-1) = (3-1)(2-1) = 2$ degrees of freedom and a right tail area of .05. Since $V_1 = 1.63 \leqslant 5.991$, conclude $C_1$.

# 23

# One-Way Analysis of Variance

Analysis of variance is a technique for studying the effects of a number of factors on the variable of interest. In one-way analysis of variance, the effect of only one factor on the variable of interest is studied.

**A. *F* Distribution.** The analysis of variance makes use of a probability distribution called the $F$ distribution. The $F$ distribution has two parameters, $\delta_2$ and $\delta_1$, which are called degrees of freedom. The $F$ distribution for $\delta_2 = 3$ degrees of freedom and $\delta_1 = 12$ degrees of freedom is shown in Figure 23.1.

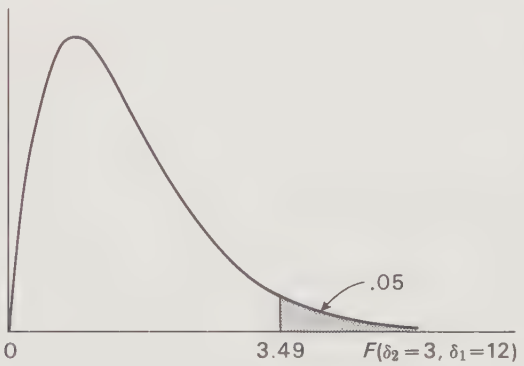



**Figure 23.1**

The mean of the $F$ distribution is $\mu = \delta_1/(\delta_1 - 2)$ for $\delta_1 > 2$. Since $F$ is a continuous random variable, probabilities relating to it correspond to areas under the curve. Table A-7 contains the $F$ values corresponding to upper tail areas of .05 and .01 for various values of $\delta_2$ and $\delta_1$.

Example 1 To illustrate the use of Table A-7, consider the $F$ value that corresponds to a right tail area of .05 for the distribution with $\delta_2 = 3$ and $\delta_1 = 12$. In Table A-7, at the intersection of the column $\delta_2 = 3$ and the row $\delta_1 = 12$, the desired $F$ value is given in lightface type, and is 3.49. $F$ values in boldface type correspond to a right tail area of .01.

**Problem 1**

Look up the $F$ values in Table A-7 for the following cases:

(a) $\delta_2 = 4, \delta_1 = 6$, right tail area = .05, $F =$_____;

(b) $\delta_2 = 8, \delta_1 = 23$, right tail area = .01, $F =$_____

**B. Double Summation.** Formulas in this section will use double summation symbols $\sum_j \sum_i$. (The use of the single summation symbol, $\Sigma$, is discussed in Section 3, Topic A.)

**Answers**

**1. (a) 4.53.**
**(b) 3.41.**

Example 2 To illustrate the use of the double summation symbol, consider the following table of $X_{ij}$ values:

**Table 23.1**

| | $j = 1$ | $j = 2$ | $j = 3$ |
|---|---|---|---|
| $i = 1$ | 4 | 3 | 1 |
| $i = 2$ | 5 | 10 | 6 |

The symbol $X_{ij}$ represents the value at the intersection of the $i$th row and the $j$th column. For instance $X_{13} = 1$, $X_{21} = 5$, and so on.

The sums of the values in each of the three columns of Table 23.1 may be written symbolically as follows:

Column one ($j = 1$): $\sum_i X_{i1} = X_{11} + X_{21} = 4 + 5 = 9.$

Column two ($j = 2$): $\sum_i X_{i2} = X_{12} + X_{22} = 3 + 10 = 13.$

Column three ($j = 3$): $\sum_i X_{i3} = X_{13} + X_{23} = 1 + 6 = 7.$

The sum of all the values in Table 23.1 is denoted by $\sum_j \sum_i X_{ij}$ and is equal to the sum of the column totals, that is:

$$\sum_j \sum_i X_{ij} = \sum_i X_{i1} + \sum_i X_{i2} + \sum_i X_{i3} = (4 + 5) + (3 + 10) + (1 + 6) = 9 + 13 + 7 = 29.$$

In general, if the subscript $i$ ranges from 1 to $m$ and the subscript $j$ ranges from 1 to $n$, then the definition of $\sum_j \sum_i X_{ij}$ is:

$$\sum_j \sum_i X_{ij} = \sum_i X_{i1} + \sum_i X_{i2} + \cdots + \sum_i X_{in},$$

where

$$\sum_i X_{i1} = X_{11} + X_{21} + \cdots + X_{m1}, \quad \sum_i X_{i2} = X_{12} + X_{22} + \cdots + X_{m2},$$

and so on.

**Problem 2**

Calculate the following sums using the $X_{ij}$ values in Table 23.1.

(a) (i) $\sum_j X_{1j} =$ _____ ; (ii) $\sum_j X_{2j} =$ _____ ;

(iii) $\sum_i \sum_j X_{ij} =$ _____ √

(b) Compare your answer in (a) (iii) with $\sum_j \sum_i X_{ij} = 29$ obtained in Example 2. What does this illustrate about the order of summation? _____

(c) (i) $\sum_i (X_{i1}^2) =$ _____ ; (ii) $\sum_j \sum_i (X_{ij}^2) =$ _____ √

(d) (i) $\sum_j (\sum_i X_{ij})^2 =$ _____ ;

(ii) $\sum_j [(\sum_i X_{ij})^2/5] =$ _____

(e) $(\sum_i \sum_j X_{ij})^2 =$ _____ √

**C. Statistical Model.**

Example 3 A home builder wishes to study the effect that type of structure has on heating costs. Three different structure types #1, #2, and #3, are to be studied. A simple random sample of four new homes is selected for each type of structure. All the homes are approximately of the same size and all heat with natural gas. Records of heating costs are obtained for the year for all 12 homes. The sample results, in dollars, are shown in Table 23.2.

**Table 23.2**

| | Type of Structure | | |
|---|---|---|---|
| Sample Home | #1 | #2 | #3 |
| 1 | 89 | 104 | 86 |
| 2 | 101 | 120 | 98 |
| 3 | 87 | 98 | 100 |
| 4 | 87 | 110 | 96 |
| Total | 364 | 432 | 380 |

Column Means: $\bar{X}_{.1} = 91$, $\bar{X}_{.2} = 108$, $\bar{X}_{.3} = 95$.

Grand Mean: $\bar{X}_{..} = (91 + 108 + 95)/3 = 98$.

The dots in the notation of the sample means indicate the subscript(s) over which the average has been taken. Thus, $\bar{X}_{.1}$ indicates the average is over all sample homes (i) for structure type #1.

The home builder wants to determine if the mean heating cost is the same for the three types of structures. This situation fits the model for a one-way analysis of variance. The three types of structures are three *treatments*, and the homes are the sampling units or *experimental units*. The random variable of interest (heating cost) is assumed to be generated by the following statistical model:

$$X_{ij} = \mu + a_j + \epsilon_{ij}.$$

Here $X_{ij}$ is the heating cost of the $i$th home with structure type $j$. The heating cost for any home is assumed to be determined by three components:

1. $\mu$ is the overall true mean cost per home of heating new homes for all three structure types.
2. $\mu + a_j = \mu_j$ is the true mean cost per home of heating new homes of structure type $j$. Thus $a_j = \mu_j - \mu$ is the deviation of the mean heating cost for homes of structure type $j$ from the mean heating cost for all three structure types. (Note that $a_j$ may be positive or negative.) The component $a_j$ frequently is called the $j$th *treatment effect*. From the definition of $a_j$, it follows that $\sum_j a_j = 0$.
3. $\epsilon_{ij}$ is a random variable representing the *experimental* or *residual error* component. It represents the effect of a multitude of uncontrolled factors on the heating cost $X_{ij}$. Note that the type of structure of the home is the only factor which is controlled in this experiment. The $\epsilon_{ij}$ are assumed to be statistically independent normally distributed random variables with mean 0 and standard deviation $\sigma$.

To illustrate how the above statistical model explains the generation of heating costs, consider the following. Suppose $\mu = \$100, a_1 = -\$2, a_2 = +\$8, a_3 = -\$6$, and $\sigma = \$8$. Let $-9, +3, -4, +2, -8, +4$ be six random $\epsilon_{ij}$ values

**Answers**

**2. (a) (i) $4 + 3 + 1 = 8$; (ii) $5 + 10 + 6 = 21$; (iii) $(4 + 3 + 1) + (5 + 10 + 6) = 29$.**
**(b) The order of summation does not affect the sum total.**
**(c) (i) $4^2 + 5^2 = 41$; (ii) $4^2 + 5^2 + 3^2 + 10^2 + 1^2 + 6^2 = 187$.**
**(d) (i) $(4 + 5)^2 + (3 + 10)^2 + (1 + 6)^2 = 299$; (ii) $(4 + 5)^2/5 + (3 + 10)^2/5 + (1 + 6)^2/5 = 299/5$.**
**(e) $(4 + 3 + 1 + 5 + 10 + 6)^2 = (29)^2 = 841$.**

from a normal distribution with mean 0 and standard deviation $\sigma = 8$. The heating cost of a home is obtained by adding together the components $\mu$, $\alpha_j$, and $\epsilon_{ij}$, as shown in Table 23.3 for selected sample homes.

**Table 23.3**

| Sample Home $i$ | Structure $j$ | Mean $\mu$ | Treatment Effect $\alpha_j$ | Residual Error $\epsilon_{ij}$ | Heating Cost $X_{ij}$ |
|---|---|---|---|---|---|
| 1 | 1 | 100 | −2 | −9 | $100 - 2 - 9 = 89$ |
| 2 | 1 | 100 | −2 | +3 | 101 |
| 1 | 2 | 100 | +8 | −4 | 104 |
| 4 | 2 | ____ | ____ | +2 | ____ |
| 1 | 3 | ____ | ____ | −8 | ____ |
| 2 | 3 | ____ | ____ | +4 | ____ |

**Problem 3**

(a) Fill in the blanks for the last three homes in Table 23.3. √

(b) If $\mu = 100$, is it consistent that $\bar{X}_{..} = 98$ in the sample study in Example 3? _____
_____
_____ √

(c) For the sample data in Example 3, what term or terms in the statistical model account for differences between:

(i) $X_{11}$ and $X_{21}$? _____
_____ ;

(ii) $\bar{X}_{.1}$ and $\bar{X}_{.2}$? _____
_____ √

**D. Sums of Squares.**

Example 4 The analysis of variance procedure involves an analysis of the variation observed in the sample data. We calculate the following three values:

$$\text{total variation} = Q = \sum_j \sum_i (X_{ij} - \bar{X}_{..})^2;$$

$$\text{within-treatments variation} = Q_1 = \sum_j \sum_i (X_{ij} - \bar{X}_{.j})^2;$$

$$\text{between-treatments variation} = Q_2 = \sum_j n(\bar{X}_{.j} - \bar{X}_{..})^2.$$

Here $\bar{X}_{.j}$ is the sample mean for the $j$th treatment, $\bar{X}_{..}$ is the overall sample mean, and $n$ is the common sample size for each treatment.

The values $Q$, $Q_1$, and $Q_2$ frequently are referred to as the total sum of squares, within-treatments sum of squares, and between-treatments sum of squares respectively. For the example here, $Q$ and $Q_2$ are computed as follows:

$Q = (89 - 98)^2 + (101 - 98)^2 + \cdots + (100 - 98)^2 + (96 - 98)^2 = 1148;$
$Q_2 = 4(91 - 98)^2 + 4(108 - 98)^2 + 4(95 - 98)^2 = 632.$

Here $n = 4$ because there is a sample of four homes for each structure type.

**Problem 4**

(a) Calculate the value of $Q_1$ for the data in Example 3. _____

(b) What is the relationship between $(Q_1 + Q_2)$ and $Q$? _____ √

**E. Analysis of Variance Table.** Often it is desired that a choice be made between the following two alternative conclusions:

$C_1$: $\mu_1 = \mu_2 = \cdots = \mu_t$;
$C_2$: not all treatment means $\mu_j$ are equal.

Here $\mu_j$ is the mean for the $j$th treatment and $t$ is the number of treatments included in the study. Since $\mu_j = \mu$ when all $\mu_j$ are equal, it follows in that case that $a_j = \mu_j - \mu = 0$. Hence, the two alternatives can be expressed in equivalent fashion as follows:

$C_1$: $a_1 = a_2 = \cdots = a_t = 0$;
$C_2$: not all treatment effects $a_j = 0$.

The test is carried out by setting up an analysis of variance table for the problem. The standard form of the one-way analysis of variance table is shown in Table 23.4.

**Table 23.4**

| Source of Variation (Source) | Sum of Squares (*SS*) | Degrees of Freedom (*df*) | Mean Square (*MS*) | Expected Mean Square [*E*(*MS*)] |
|---|---|---|---|---|
| Between-treatments | $Q_2$ | $t - 1$ | $MS_2 = Q_2/(t-1)$ | $E(MS_2) = \sigma^2 + n(a_1^2 + \cdots + a_t^2)/(t-1)$ |
| Within-treatments | $Q_1$ | $t(n-1)$ | $MS_1 = Q_1/t(n-1)$ | $E(MS_1) = \sigma^2$ |
| Total | $Q$ | $nt - 1$ | | |

With each sum of squares is associated a parameter value called degrees of freedom. For instance, $Q_1$ has $t(n-1)$ degrees of freedom. Dividing a sum of squares value by its degrees of freedom yields the corresponding mean square value. For instance, $Q_1/t(n-1)$ is the mean square value for within-treatments variation, denoted by $MS_1$. The expected values of the mean square terms are shown in the column on the extreme right. For example, $E(MS_1)$ is the expected value of $MS_1$.

**Problem 5**

For Example 3, $Q_1 = 516$, $Q_2 = 632$, $Q = 1148$, $t = 3$, and $n = 4$. Following the standard form of the one-way analysis of variance table (Table 23.4), we obtain:

**Table 23.5**

| Source | *SS* | *df* | *MS* | *E*(*MS*) |
|---|---|---|---|---|
| Between-treatments | 632 | 2 | 316 | $\sigma^2 + 4(a_1^2 + a_2^2 + a_3^2)/2$ |
| Within-treatments | ___ | 9 | ___ | $\sigma^2$ |
| Total | 1148 | ___ | | |

Answers

3. (a)

| $i$ | $j$ | $\mu$ | $a_j$ | $\epsilon_{ij}$ | $X_{ij}$ |
|---|---|---|---|---|---|
| 4 | 2 | 100 | +8 | +2 | 110 |
| 1 | 3 | 100 | −6 | −8 | 86 |
| 2 | 3 | 100 | −6 | +4 | 98 |

(b) The random error component in the model leads to sample estimates that vary to some extent from the parameter values. Sometimes it will lead to a sample estimate which is higher, sometimes it will lead to a sample estimate which is lower. In this case, the sample estimate is lower.

(c) (i) Random error component varies for the two observations.

(ii) Random error component varies from observation to observation, and treatment effect may differ for the two treatments.

4. (a) $Q_1 = (89-91)^2 + (101-91)^2 + \cdots + (100-95)^2 + (96-95)^2 = 516$.

(b) $Q_1 + Q_2 = Q$. In this example, $516 + 632 = 1148$.

(a) State the alternative conclusions. $C_1$: _____; $C_2$: _____ √

(b) With the information given, fill in the blank spaces in Table 23.5.

(c) Do the between- and within-treatments degrees of freedom add to the degrees of freedom for total variation?

_____

(d) If $C_1$ holds, what is the expected mean square value for between-treatments variation? _____

_____ √

(e) If $C_1$ holds, how do $E(MS_1)$ and $E(MS_2)$ compare? _____

(f) If $C_1$ does not hold, is $E(MS_1)$ or $E(MS_2)$ larger? (*Hint*: Suppose $a_1 = -1$, $a_2 = +10$, $a_3 = -9$, what is $E(MS_2)$?) _____ √

**F. Test Statistic and Appropriate Decision Rule.** Since it was seen in Problem 5 that when $C_1$ holds, $E(MS_1) = E(MS_2)$, one would expect $MS_2$ and $MS_1$ to be of about equal magnitude in that case. On the other hand, if $C_2$ holds, that is, if not all $a_1, a_2, \ldots, a_I$ are zero, $E(MS_2)$ will be larger than $E(MS_1)$, and one would expect $MS_2$ to be greater than $MS_1$. Consequently, we decide whether to conclude $C_1$ or $C_2$ by comparing $MS_2$ and $MS_1$. Specifically, we use the test statistic $V_2 = MS_2/MS_1$ for the test. The appropriate decision rule is a one-sided upper-tail one: If $V_2 \leqslant A$, conclude $C_1$; if $V_2 > A$, conclude $C_2$. Here $A$ is a specified action limit.

To control the risk of making a Type I error, we are interested in the distribution of the test statistic $V_2$ if $C_1$ holds. Statistical theory shows that if $C_1$ holds, $V_2$ is distributed according to an $F$ distribution with $\delta_2 = t - 1$ and $\delta_1 = t(n-1)$, the degrees of freedom associated with $MS_2$ and $MS_1$ respectively. Thus, the action limit of the decision rule is an $F$ value corresponding to the level of significance of the test (risk of Type I error).

Example 5 In our study of heating costs, assume that the level of significance is specified to be .05. If $C_1$ holds, $V_2$ is distributed according to the $F$ distribution with $\delta_2 = t - 1 = 3 - 1 = 2$ and $\delta_1 = 3(4-1) = 9$. For this $F$ distribution, the $F$ value corresponding to .05 in the right tail is 4.26. Hence, the action limit of the decision rule is $A = 4.26$ as shown in Figure 23.2.

The decision rule then is: If $V_2 \leqslant 4.26$, conclude $C_1$; if $V_2 > 4.26$, conclude $C_2$.

**Problem 6**

(a) Using the results in Problem 5, calculate the test statistic $V_2$. _____

(b) Using the decision rule in Example 5, what conclusion should be reached? _____

(c) Express in words the conclusion reached. _____

_____

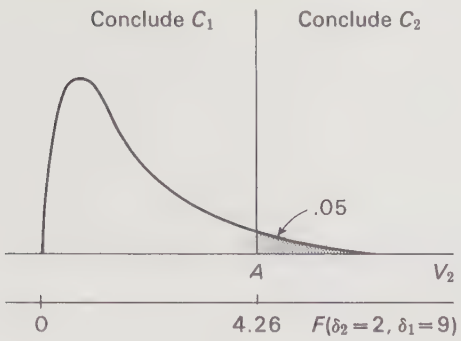



**Figure 23.2**

(d) Suppose 4 structure types had been studied, and 5 sample homes used for each structure type. What would be the appropriate decision rule, if the level of significance is to be .01? _____ √

**Problem 7**

The variance of the error term $\epsilon_{ij}$, denoted by $\sigma^2$, measures the residual error or inherent variability, and reflects the effects of factors that are not controlled in the study.

(a) Refer to Problem 5. Estimate $\sigma^2$ for the heating cost study. _____ Is this estimator unbiased? _____

(b) Would there be any advantage in reducing the residual error? _____ √

Answers

5. (a) $C_1$: $a_1 = a_2 = a_3 = 0$.
$C_2$: not all $a_j = 0$.

(b)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Between-treatments | 632 | 2 | 316 |
| Within-treatments | 516 | 9 | 57.33 |
| Total | 1148 | 11 | |

(c) Yes, $2 + 9 = 11$.

(d) $E(MS_2) = \sigma^2 + 4(a_1^2 + a_2^2 + a_3^2)/2$. If $C_1$ holds, then $a_1 = a_2 = a_3 = 0$ and $E(MS_2) = \sigma^2 + 4(0^2 + 0^2 + 0^2)/2 = \sigma^2$.

(e) $E(MS_1) = E(MS_2) = \sigma^2$.

(f) $E(MS_2)$ is larger because the term $4(a_1^2 + a_2^2 + a_3^2)/2$ is positive. For instance, for the treatment effects given in the hint:

$$E(MS_2) = \sigma^2 + 4\,[(-1)^2 + (+10)^2 + (-9)^2]\,/2 = \sigma^2 + 364.$$

6. (a) $V_2 = 316/57.33 = 5.51$.

(b) Conclude $C_2$.

(c) The mean heating cost is not the same for all three structure types, that is, not all treatment effects $(a_j)$ equal zero.

(d) Decision rule: If $V_2 \leqslant 5.29$, conclude $C_1$; if $V_2 > 5.29$, conclude $C_2$. Here 5.29 is the $F$ value corresponding to .01 in the right tail with $\delta_2 = 4 - 1 = 3$ and $\delta_1 = 4(5 - 1) = 16$.

7. (a) $MS_1 = 57.33$ is an estimate of $\sigma^2$. $MS_1$ is an unbiased estimator of $\sigma^2$ because $E(MS_1) = \sigma^2$.

(b) **For a given sample size and specified level of significance, reducing the experimental error decreases Type II error probabilities. This explains why efforts should be made by the researcher to reduce experimental error in his study.**

## REVIEW PROBLEMS

1. A tire manufacturer subjected three sets of three tires each to wear tests in a laboratory. The three sets of tires were constructed of experimental material A, experimental material B, and conventional material C, respectively. The results, in thousands of miles of wear, are:

| Sample Tire | Material A | Material B | Material C |
|---|---|---|---|
| #1 | 24 | 40 | 25 |
| #2 | 24 | 39 | 32 |
| #3 | 30 | 44 | 24 |

The manufacturer wishes to test whether or not the mean wear (in thousands of miles) is the same for tires made from the three materials.

(a) Specify the alternative conclusions. $C_1$: _____; $C_2$: _____ _____ √

(b) Set up an analysis of variance table (including the $E(MS)$ column), and obtain the value of $V_2$. √

(c) What is the distribution of $V_2$ if the mean wear is the same for the three materials? _____ _____

(d) Determine the appropriate decision rule if the risk of a Type I error is to be controlled at .05. _____ _____

What conclusion should be reached? _____

(e) Estimate the inherent sampling variability in this experiment. _____

(f) Provide a point estimate of the mean number of miles of extra wear that can be obtained from a tire made from material B compared with a tire made from material C. _____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a) $C_1$: $a_A = a_B = a_C = 0$;
$C_2$: not all treatment effects are zero.

(b)

| Source | *SS* | *df* | *MS* | *E*(*MS*) |
|---|---|---|---|---|
| Between-treatments | 422 | 2 | 211 | $\sigma^2 + 3(a_A^2 + a_B^2 + a_C^2)/2$ |
| Within-treatments | 76 | 6 | 12.67 | $\sigma^2$ |
| Total | 498 | 8 | | |

$V_2 = 211/12.67 = 16.65$.

(c) $F$ distribution with $\delta_2 = 2$ and $\delta_1 = 6$.

(d) Decision rule: If $V_2 \leq 5.14$, conclude $C_1$; if $V_2 > 5.14$, conclude $C_2$.
Since $V_2 = 16.65 > 5.14$, conclude $C_2$.

(e) $MS_1 = 12.67$.

(f) $\bar{X}_{.B} - \bar{X}_{.C} = 41 - 27 = 14$ thousand miles is a point estimate of the mean number of miles of extra wear that can be obtained from material B compared with material C.

# 24

# Two-Way Analysis of Variance

Two-way analysis of variance is used to study the effects of two factors on the variable of interest, including the interactions of the two factors.

**A. Sums of Squares.**

Example 1 Consider the following modification of Example 3 from Section 23. A home builder wishes to study the effects of type of structure ($A_1, A_2, A_3$) and type of insulation ($B_1, B_2$) on home heating costs. A simple random sample of two new homes is obtained for each combination of type of structure and type of insulation. All the homes are approximately of the same size and all heat with natural gas. Records of heating costs are obtained for the current year for the twelve homes. The sample results are given in Table 24.1.

**Table 24.1**

| Sample Home | Structure and Insulation Combination | | | | | |
|---|---|---|---|---|---|---|
| | $A_1B_1$ | $A_2B_1$ | $A_3B_1$ | $A_1B_2$ | $A_2B_2$ | $A_3B_2$ |
| #1 | 89 | 120 | 100 | 87 | 98 | 86 |
| #2 | 101 | 110 | 98 | 87 | 104 | 96 |
| Total | 190 | 230 | 198 | 174 | 202 | 182 |
| Means | $\bar{X}_{11} = 95$ | $\bar{X}_{21} = 115$ | $\bar{X}_{31} = 99$ | $\bar{X}_{12} = 87$ | $\bar{X}_{22} = 101$ | $\bar{X}_{32} = 91$ |

To study whether or not the mean heating cost is the same for the six treatments (structure and insulation combinations), we proceed with a one-way analysis of variance as described in Section 23:

**Table 24.2**

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Between-treatments | ____ | ____ | ____ |
| Within-treatments | 192 | ____ | ____ |
| Total | 1148 | 11 | |

**Problem 1**

(a) Complete Table 24.2 (*Hint*: $Q = Q_1 + Q_2$.)

(b) Test whether all the six treatment effects $a_j$ are equal to zero. Control the probability of a Type I error (level of significance) at .05.

(c) Suppose the probability of a Type I error should have been controlled at .10. Even though Table A-7 does not contain $F$ values for that error probability, can you tell what the appropriate conclusion should be in that case? Explain. _____

_____ √

Example 2 Consider the study described in Example 1. In two-way analysis of variance, we are interested in analyzing the contribution which the effects of factor $A$ (structure), factor $B$ (insulation), and interaction $AB$ (interaction of structure and insulation) make to the explanation of variation in heating costs. Letting $Q_A$, $Q_B$, and $Q_{AB}$ denote the sums of squares for factor $A$, factor $B$, and interaction $AB$ effects respectively, it can be shown that between-treatments sum of squares ($Q_2$) can be partitioned as follows:

$$Q_2 = Q_A + Q_B + Q_{AB}.$$

The sums of squares, $Q_A$, $Q_B$, and $Q_{AB}$, are calculated using the following formulas:

$$Q_A = bn \sum_i (\bar{X}_{i.} - \bar{X}_{..})^2;$$

$$Q_B = an \sum_j (\bar{X}_{.j} - \bar{X}_{..})^2;$$

$$Q_{AB} = n \sum_j \sum_i (\bar{X}_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2.$$

In these formulas,

$a$ = number of classes of factor $A$;
$b$ = number of classes of factor $B$;
$n$ = number of sample observations in each treatment ($AB$ combination);
$\bar{X}_{ij}$ = sample mean for the $A_iB_j$ treatment;
$\bar{X}_{i.}$ = sample mean for all observations in $A_i$;
$\bar{X}_{.j}$ = sample mean for all observations in $B_j$;
$\bar{X}_{..}$ = sample mean for all observations.

In this example, $a = 3$, $b = 2$, and $n = 2$. The $\bar{X}_{ij}$ treatment means were given in Table 24.1, and are restated in Table 24.3.

**Table 24.3**

| | | Insulation | | |
|---|---|---|---|---|
| | | $B_1$ | $B_2$ | Row Means |
| Structure Type | $A_1$ | 95 | 87 | $\bar{X}_{1.} = 91$ |
| | $A_2$ | 115 | 101 | $\bar{X}_{2.} = 108$ |
| | $A_3$ | 99 | 91 | $\bar{X}_{3.} = 95$ |
| Column Means | | $\bar{X}_{.1} = 103$ | $\bar{X}_{.2} = 93$ | $\bar{X}_{..} = 98$ |

For example, 95 is the sample mean for the $A_1B_1$ treatment combination. The sample means, $\bar{X}_{i.}$ for the $A$ factor classes, $\bar{X}_{.j}$ for the $B$ factor classes, and $\bar{X}_{..}$, the overall mean, are shown in the margins of Table 24.3. For instance, $\bar{X}_{1.} = (95 + 87)/2 = 91$, $\bar{X}_{.2} = (87 + 101 + 91)/3 = 93$, and $\bar{X}_{..} = (103 + 93)/2 = (91 + 108 + 95)/3 = 98$.

Substituting these values into the formulas for $Q_A$, $Q_B$, and $Q_{AB}$, we obtain:

$$Q_A = (2)(2)[(91 - 98)^2 + (108 - 98)^2 + (95 - 98)^2] = 632;$$

$$Q_B = (3)(2)[(103 - 98)^2 + (93 - 98)^2] = 300;$$

$$Q_{AB} = (2)[(95 - 91 - 103 + 98)^2 + \cdots + (91 - 95 - 93 + 98)^2] = _____.$$

**Problem 2**

(a) Obtain the value of $Q_{AB}$ in Example 2. _____

(b) In the answer to Problem 1, we found that $Q_2 = 956$. Verify that $Q_2 = Q_A + Q_B + Q_{AB}$. _____ _____ Note that this relation provides you with a check on your computation.√

**B. Model.** The statistical model underlying two-way analysis of variance is:

$$X_{ijk} = \mu + a_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}.$$

Here $X_{ijk}$ is the value of the $k$th sample observation subjected to treatment $A_iB_j$, $\mu$ is the overall mean, $a_i$ is the effect of factor $A_i$, $\beta_j$ is the effect of factor $B_j$, $\gamma_{ij}$ is the interaction effect of factor $A_i$ and factor $B_j$, and $\epsilon_{ijk}$ is the experimental error for the $k$th sample observation subjected to treatment $A_iB_j$. The model components, $\mu$, $a_i$, $\beta_j$, and $\gamma_{ij}$, are constants with the properties:

$$\Sigma a_i = \Sigma\beta_j = \sum_i \gamma_{ij} = \sum_j \gamma_{ij} = 0.$$

The sum $\mu + a_i + \beta_j + \gamma_{ij}$ equals $\mu_{ij}$, the mean response for the $A_iB_j$ treatment. The error terms, $\epsilon_{ijk}$, are assumed to be statistically independent normally distributed random variables with mean 0 and standard deviation $\sigma$.

We illustrate the meaning of the terms in the model with reference to our example:

$\mu$: overall true mean heating cost of a home for all three structure and two insulation types.

$a_1$: change from overall mean heating cost $\mu$ which results from the fact that the home has structure $A_1$ (ignoring type of insulation).

$\beta_2$: change from overall mean heating cost $\mu$ which results from the fact that the home has insulation $B_2$ (ignoring type of structure).

$\gamma_{12}$: change from mean heating cost $\mu + a_1 + \beta_2$ which would be expected when structure $A_1$ and insulation $B_2$ are used if any interaction effects are ignored.

$\epsilon_{121}$: random effect which prevails in the first home of the $A_1B_2$ treatment group as the result of factors which are not controlled in the experiment.

Three sets of alternative conclusions that are tested frequently in two-way analysis of variance are:

$C_1$: $a_1 = a_2 = \cdots = a_a = 0$;
$C_2$: not all $a_i$ effects are zero.

$C_1$: $\beta_1 = \beta_2 = \cdots = \beta_b = 0$;
$C_2$: not all $\beta_j$ effects are zero.

$C_1$: $\gamma_{11} = \gamma_{12} = \cdots = \gamma_{ab} = 0$;
$C_2$: not all $\gamma_{ij}$ effects are zero.

**Problem 3**

For the heating cost study in Example 1, state the three sets of alternative conclusions:

$C_1$: _____ $C_1$: _____ $C_1$: _____
$C_2$: _____ $C_2$: _____ $C_2$: _____ √

Answers

1. (a)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Between-treatments | 956 | 5 | 191.2 |
| Within-treatments | 192 | 6 | 32 |
| Total | 1148 | 11 | |

(b) $C_1$: all $\alpha_j = 0$;
$C_2$: not all $\alpha_j = 0$.
The $F$ value for $\delta_2 = 5$ and $\delta_1 = 6$ and a right tail area of .05 is 4.39. The decision rule is: If $V_2 \leqslant 4.39$, conclude $C_1$; if $V_2 > 4.39$, conclude $C_2$.
$V_2 = MS_2/MS_1 = 191.2/32 = 5.98$. Since $V_2 = 5.98 > 4.39$, conclude $C_2$.

(c) Since the action limit associated with the level of significance .10 would be less than (to the left of) the action limit used in (b), the sample result would still lead to conclusion $C_2$.

2. (a) $$Q_{AB} = 2[(95 - 91 - 103 + 98)^2 + (87 - 91 - 93 + 98)^2 + (115 - 108 - 103 + 98)^2 + (101 - 108 - 93 + 98)^2 + (99 - 95 - 103 + 98)^2 + (91 - 95 - 93 + 98)^2] = 24.$$

(b) $632 + 300 + 24 = 956$.

3. $C_1$: $\alpha_1 = \alpha_2 = \alpha_3 = 0$; $C_1$: $\beta_1 = \beta_2 = 0$;
$C_2$: not all $\alpha_j = 0$; $C_2$: not all $\beta_j = 0$.
$C_1$: $\gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = \gamma_{31} = \gamma_{32} = 0$;
$C_2$: not all $\gamma_{ij} = 0$.

C. **Two-Way Analysis of Variance Table.** The tests of the alternative conclusions are carried out by setting up an analysis of variance table. The standard form of the two-way analysis of variance table is shown in Table 24.4.

**Table 24.4**

| Source | *SS* | *df* | *MS* | *E*(*MS*) |
|---|---|---|---|---|
| Between-treatments | $Q_2$ | $ab - 1$ | $MS_2$ | $\sigma^2 + \frac{n}{ab-1}\Sigma\Sigma(\alpha_j + \beta_j + \gamma_{ij})^2$ |
| Factor $A$ | $Q_A$ | $a - 1$ | $MS_A$ | $\sigma^2 + \frac{nb}{a-1}\Sigma\alpha_i^2$ |
| Factor $B$ | $Q_B$ | $b - 1$ | $MS_B$ | $\sigma^2 + \frac{na}{b-1}\Sigma\beta_j^2$ |
| Interaction $AB$ | $Q_{AB}$ | $(a-1)(b-1)$ | $MS_{AB}$ | $\sigma^2 + \frac{n}{(a-1)(b-1)}\Sigma\Sigma\gamma_{ij}^2$ |
| Within-treatments | $Q_1$ | $ab(n-1)$ | $MS_1$ | $\sigma^2$ |
| Total | $Q$ | $nab - 1$ | | |

The expected values of the mean square terms, $E(MS)$, are shown on the extreme right of the analysis of variance table.

**Example 3** We obtain the following table for the heating cost study:

**Table 24.5**

| Source | *SS* | *df* | *MS* | *E*(*MS*) |
|---|---|---|---|---|
| Between-treatments | ____ | ____ | 191.2 | $\sigma^2 + \frac{2}{5}\Sigma\Sigma(\alpha_i + \beta_j + \gamma_{ij})^2$ |
| Factor *A* | 632 | 2 | ____ | $\sigma^2 + 2\,\Sigma\,\alpha_i^2$ |
| Factor *B* | ____ | ____ | ____ | $\sigma^2 + 6\,\Sigma\,\beta_j^2$ |
| Interaction *AB* | 24 | 2 | ____ | $\sigma^2 + \Sigma\Sigma\,\gamma_{ij}^2$ |
| Within-treatments | 192 | ____ | ____ | $\sigma^2$ |
| Total | 1148 | 11 | | |

**Problem 4**

(a) Fill in the blank spaces in Table 24.5. (*Hint*: Use $Q_2 = Q_A + Q_B + Q_{AB}$ as a check.)
(b) Are the degrees of freedom associated with the effects of factor *A*, factor *B*, and *AB* interactions additive? ____ Are the corresponding mean squares additive? ____ √
(c) If there are no factor *A* effects, i.e., if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, what is $E(MS_A)$? ____ How does this compare with $E(\mathrm{MS}_1)$? ____
(d) If $\alpha_1 = 3$, $\alpha_2 = 7$, and $\alpha_3 = -10$, what is $E(MS_A)$? ____ How does this compare with $E(MS_1)$? ____
(e) Do large or small values of $MS_A/MS_1$ suggest that factor *A* effects are present? ____ √

**D. Appropriate Decision Rule.** The presence of factor *A* effects, factor *B* effects, and *AB* interaction effects may be tested in the same manner that the presence of treatment effects was tested in the one-way analysis of variance model. Shown in Table 24.6, for each test, are the appropriate test statistic, decision rule, and the sampling distribution of the test statistic if $C_1$ holds.

**Table 24.6**

| Alternative Conclusions | Test Statistic | Decision Rule | Sampling Distribution if $C_1$ Holds |
|---|---|---|---|
| $C_1$: $\alpha_1 = \cdots = \alpha_a = 0$ | $V_A = \frac{MS_A}{MS_1}$ | If $V_A \leqslant A$, conclude $C_1$ | $F(a-1, ab[n-1])$ |
| $C_2$: not all $\alpha_i = 0$ | | If $V_A > A$, conclude $C_2$ | |
| $C_1$: $\beta_1 = \cdots = \beta_b = 0$ | $V_B = \frac{MS_B}{MS_1}$ | If $V_B \leqslant A$, conclude $C_1$ | $F(b-1, ab[n-1])$ |
| $C_2$: not all $\beta_j = 0$ | | If $V_B > A$, conclude $C_2$ | |
| $C_1$: $\gamma_{11} = \cdots = \gamma_{ab} = 0$ | $V_{AB} = \frac{MS_{AB}}{MS_1}$ | If $V_{AB} \leqslant A$, conclude $C_1$ | $F([a-1][b-1], ab[n-1])$ |
| $C_2$: not all $\gamma_{ij} = 0$ | | If $V_{AB} > A$, conclude $C_2$ | |

**Answers**

4. (a)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Between-treatments | 956 | 5 | 191.2 |
| Factor *A* | 632 | 2 | 316 |
| Factor *B* | 300 | 1 | 300 |
| Interaction *AB* | 24 | 2 | 12 |
| Within-treatments | 192 | 6 | 32 |
| Total | 1148 | 11 | |

**(b) An inspection of the table indicates that the degrees of freedom are additive (2 + 1 + 2 = 5) but the mean squares are not additive (316 + 300 + 12 ≠ 191.2).**

**(c)** $E(MS_A) = \sigma^2 + 2 \Sigma \alpha_i^2$. If $\alpha_1 = \alpha_2 = \alpha_3 = 0$, then:

$$E(MS_A) = \sigma^2 + 2(0^2 + 0^2 + 0^2) = \sigma^2.$$

In this case, $E(MS_A) = E(MS_1) = \sigma^2$.

**(d)** If $\alpha_1 = 3$, $\alpha_2 = 7$, and $\alpha_3 = -10$, then:

$$E(MS_A) = \sigma^2 + 2\,[(3)^2 + (7)^2 + (-10)^2] = \sigma^2 + 316.$$

In this case, $E(MS_A) > E(MS_1)$ because $\sigma^2 + 316 > \sigma^2$.

**(e)** Large values of $MS_A/MS_1$ suggest that factor *A* effects are present, that is, that not all $\alpha_i = 0$.

Example 4 Controlling the Type I error probability at .05, the test results for the presence of structure effects (factor *A*) are summarized in the first line of Table 24.7.

Table 24.7

| Test | Value of Test Statistic | Action Limit at .05 Level | Conclusion |
|---|---|---|---|
| Factor *A* | $V_A = 316/32 = 9.88$ | $F(2,6) = 5.14$ | $C_2$: not all $\alpha_i = 0$ |
| Factor *B* | _____ | _____ | _____ |
| Interaction *AB* | _____ | _____ | _____ |

**Problem 5**

(a) Complete the last two lines of Table 24.7 to test for the presence of factor *B* effects and *AB* interactions. In each case, control the probability of a Type I error at .05.

(b) In words, interpret the three conclusions reached in the tests. _____

_____

_____

_____

_____√

**E. Point Estimators of Effects and Inherent Variability.** Point estimators of $\mu$, $\alpha_i$, $\beta_j$, $\gamma_{ij}$, and $\sigma^2$ are shown below:

**Table 24.8**

| Parameter | Point Estimator | Parameter | Point Estimator |
|---|---|---|---|
| $\mu$ | $\overline{X}_{..}$ | $\gamma_{ij}$ | $\overline{X}_{ij} - \overline{X}_{i.} - \overline{X}_{.j} + \overline{X}_{..}$ |
| $\alpha_j$ | $\overline{X}_{i.} - \overline{X}_{..}$ | $\sigma^2$ | $MS_1$ |
| $\beta_j$ | $\overline{X}_{.j} - \overline{X}_{..}$ | | |

All of these point estimators are unbiased.

Example 5 To illustrate the calculation of point estimates of parameters, observe that $\overline{X}_{..} = 98$ for the heating cost study. Thus, $\mu$, the overall mean cost of heating a home for all structure and insulation types is estimated to be \$98.

**Problem 6**

For the heating cost study, give point estimates of the following:
(a) $\alpha_1$ ____; (b) $\beta_2$ ____; (c) the mean cost of heating a home with structure type $A_3$ ____;
(d) $\gamma_{22}$ ____; (e) the mean cost of heating a home with structure $A_1$ and insulation $B_2$ ____;
(f) experimental error (inherent variability) ____ √

## REVIEW PROBLEMS

1. A firm is introducing a new product. It wishes to study the effect on sales of using two different counter displays, $A_1$ and $A_2$, and two different prices, $B_1$ and $B_2$, in promoting the product. For each combination of display and price, three stores are selected at random to sell the product. The unit sales in each of the stores are recorded for a trial period. The results of the experiment are given in Table 24.9.

**Table 24.9**

| Sample Store | Display and Price Combination | | | |
|---|---|---|---|---|
| | $A_1B_1$ | $A_1B_2$ | $A_2B_1$ | $A_2B_2$ |
| #1 | 64 | 72 | 60 | 88 |
| #2 | 84 | 70 | 54 | 84 |
| #3 | 56 | 74 | 60 | 86 |

(a) Set up and complete the two-way analysis of variance table for this experiment. √

(Review Problem 1 continued on page 211.)

Answers

5. (a)

| Test | Value of Test Statistic | Action Limit at .05 Level | Conclusion |
|---|---|---|---|
| Factor $B$ | $V_B = 300/32 = 9.38$ | $F(1,6) = 5.99$ | $C_2$: not all $\beta_j = 0$ |
| Interaction $AB$ | $V_{AB} = 12/32 = .38$ | $F(2,6) = 5.14$ | $C_1$: $\gamma_{11} = \cdots = \gamma_{32} = 0$ |

(b) Both type of structure and type of insulation significantly affect heating costs. Further, these two factors affect heating costs independently since the interaction effects equal zero and therefore type of structure and type of insulation do not interact.

6.

| Parameter | Point Estimate |
|---|---|
| (a) $\alpha_1$ | $\bar{X}_{1.} - \bar{X}_{..} = 91 - 98 = -\$7$ |
| (b) $\beta_2$ | $\bar{X}_{.2} - \bar{X}_{..} = 93 - 98 = -\$5$ |
| (c) $\mu + \alpha_3$ | $\bar{X}_{3.} = \$95$ |
| (d) $\gamma_{22}$ | $\bar{X}_{22} - \bar{X}_{2.} - \bar{X}_{.2} + \bar{X}_{..} = 101 - 108 - 93 + 98 = -\$2$ |
| (e) $\mu + \alpha_1 + \beta_2 + \gamma_{12}$ | $\bar{X}_{12} = \$87$ |
| (f) $\sigma^2$ | $MS_1 = 32$ |

## ANSWERS TO REVIEW PROBLEMS

1. (a)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Between-treatments | 1212 | 3 | 404 |
| Factor $A$ | 12 | 1 | 12 |
| Factor $B$ | 768 | 1 | 768 |
| Interaction $AB$ | 432 | 1 | 432 |
| Within-treatments | 456 | 8 | 57 |
| Total | 1668 | 11 | |

The treatment means $\bar{X}_{ij}$ and other averages are:

| | $B_1$ | $B_2$ | $\bar{X}_{i.}$ |
|---|---|---|---|
| $A_1$ | 68 | 72 | 70 |
| $A_2$ | 58 | 86 | 72 |
| $\bar{X}_{.j}$ | 63 | 79 | $\bar{X}_{..} = 71$ |

(b) Test whether or not: (i) display effects are present; (ii) price effects are present; (iii) display and price interaction effects are present. In each of your tests, control the probability of a Type I error at .05.√

(c) Interpret the conclusions of the three tests.√

(d) In the experiment, which display and price combination led to the highest average sales volume? _____

(e) For this combination, give a point estimate of: (i) display effect _____, (ii) price effect _____, (iii) interaction effect _____ √

(f) (i) Give a point estimate of the inherent variability in this experiment. _____

(ii) In conducting this experiment, how might the firm reduce the inherent variability? _____

_____

_____√

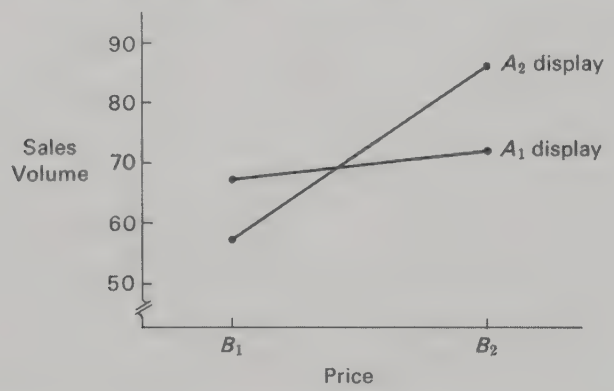
## ANSWERS TO REVIEW PROBLEMS

1. cont'd.

(b)

| Test | | Alternative Conclusions | Test Statistic | Action Limit at .05 Level | Conclusion |
|---|---|---|---|---|---|
| Display effects | $C_1$: | $a_1 = a_2 = 0$ | $V_A = 12/57 = .21$ | $F(1,8) = 5.32$ | $C_1$: $a_1 = a_2 = 0$ |
| | $C_2$: | not all $a_i = 0$ | | | |
| Price effects | $C_1$: | $\beta_1 = \beta_2 = 0$ | $V_B = 768/57 = 13.47$ | $F(1,8) = 5.32$ | $C_2$: not all $\beta_j = 0$ |
| | $C_2$: | not all $\beta_j = 0$ | | | |
| Interaction | $C_1$: | $\gamma_{11} = \gamma_{12} = \gamma_{21} = \gamma_{22} = 0$ | $V_{AB} = 432/57 = 7.58$ | $F(1,8) = 5.32$ | $C_2$: not all $\gamma_{ij} = 0$ |
| | $C_2$: | not all $\gamma_{ij} = 0$ | | | |

(c) The conclusions are that price affects sales volume, and that price and display effects interact. The interaction of display and price can be seen clearly in the following response graph:




Note how sensitive sales volume is to a price change when display $A_2$ is used, while with display $A_1$, sales volume is quite insensitive to a price change.

(d) Display $A_2$ and price $B_2$ led to the highest average sales volume in the experiment.

(e)

| | Parameter | Point Estimate |
|---|---|---|
| (i) | display effect ($a_2$) | $\bar{X}_{2\cdot} - \bar{X}_{\cdot\cdot} = 72 - 71 = +1$ |
| (ii) | price effect ($\beta_2$) | $\bar{X}_{\cdot 2} - \bar{X}_{\cdot\cdot} = 79 - 71 = +8$ |
| (iii) | interaction effect ($\gamma_{22}$) | $\bar{X}_{22} - \bar{X}_{2\cdot} - \bar{X}_{\cdot 2} + \bar{X}_{\cdot\cdot} = 86 - 72 - 79 + 71 = +6$ |

(f) (i) A point estimate of $\sigma^2$ is $MS_1 = 57$.

(ii) The firm can reduce inherent variability in the experiment by controlling factors affecting the sales volume presently not controlled, such as differences in store size, location, and clientele, position of displays in the store, and so on.

# 25

# Inferences Concerning Variances

In this section, confidence intervals and decision rules for population variances will be constructed. We assume in all cases that the relevant populations are normal.

**A. Confidence Interval for a Population Variance.** Readers who are not familiar with the $\chi^2$ distribution should refer first to Topic A, Section 21.

Statistical theory shows that the ratio $(n-1)s^2/\sigma^2$ has a $\chi^2$ distribution with $n-1$ degrees of freedom when sampling from a normal population with variance $\sigma^2$. Here $s^2 = \Sigma(X - \overline{X})^2/(n-1)$. The $(1-\alpha)100$ per cent confidence interval for $\sigma^2$ is given by:

$$\frac{(n-1)s^2}{B} \leqslant \sigma^2 \leqslant \frac{(n-1)s^2}{A},$$

where $A$ and $B$ denote respectively the $\chi^2$ values corresponding to left tail and right tail areas of $\alpha/2$ each in the $\chi^2$ distribution with $n-1$ degrees of freedom.

**Example 1** A manufacturer wishes to construct a 98 per cent confidence interval for the variance of the adhesive strength of his tape because he is concerned about the consistency of the tape's strength. A small strip of tape was taken from each of six rolls selected at random, and the strength measured. The sample variance was calculated to be $s^2 = .7$. Assume the adhesive strengths are normally distributed.

For a 98 per cent confidence coefficient, we have $\alpha/2 = .01$. Table A-6 gives, for $\alpha/2 = .01$ and $n - 1 = 6 - 1 = 5$ degrees of freedom, $A = .554$ and $B = 15.086$. The 98 per cent confidence interval for $\sigma^2$ therefore is:

$$\frac{(6-1)(.7)}{15.086} \leqslant \sigma^2 \leqslant \frac{(6-1)(.7)}{.554} \quad \text{or} \quad .23 \leqslant \sigma^2 \leqslant 6.32.$$

We thus conclude that the population variance $\sigma^2$ is between .23 and 6.32 (98 per cent confidence coefficient).

**Problem 1**

The number of units of a product which are demanded each month follows a normal distribution. The sample variance of demand for a random sample of twelve months is $s^2 = 14$.

(a) Construct a 90 per cent confidence interval for the population variance of monthly demand. _____ _____ √

(b) Using your results in (a), what is the 90 per cent confidence interval for the population standard deviation $\sigma$? _____ √

**B. Construction of Decision Rule Concerning Population Variance.** We consider the construction of a decision rule for choosing between the following alternative conclusions:

$$C_1: \quad \sigma^2 = \sigma_0^2; \quad C_2: \quad \sigma^2 \neq \sigma_0^2.$$

**Answers**

**1. (a)** $s^2 = 14$; $a/2 = .05$; $n - 1 = 12 - 1 = 11$; $A = 4.575$; $B = 19.675$.

The confidence interval is: $(11)(14)/19.675 \leqslant \sigma^2 \leqslant (11)(14)/4.575$ which gives: $7.83 \leqslant \sigma^2 \leqslant 33.7$.

**(b)** $7.83 \leqslant \sigma^2 \leqslant 33.7$ implies $\sqrt{7.83} \leqslant \sigma \leqslant \sqrt{33.7}$ or $2.80 \leqslant \sigma \leqslant 5.81$.

The value $\sigma_0^2$ is a specified indifference point, standard, level from past experience, or the like. The construction of decision rules for one-sided alternatives follows in a straightforward manner.

Example 2 Refer to Example 1, where $s^2 = .7$ for $n = 6$. Suppose experience indicates that in the past the population variance of the adhesive strength was 2.0. The manufacturer wishes to know whether or not $\sigma^2 = 2.0$ for the current production process. Thus the alternative conclusions are: $C_1$: $\sigma^2 = 2.0$; $C_2$: $\sigma^2 \neq 2.0$. Assume the manufacturer wishes to control the probability of a Type I error at .02.

The appropriate test statistic is $V_3 = (n - 1)s^2/\sigma_0^2$. In this example, $V_3 = (6 - 1)(.7)/2.0 = 1.75$. The decision rule for controlling the risk of making a Type I error (level of significance) at $a$ has the following form: If $A \leqslant V_3 \leqslant B$, conclude $C_1$; if $V_3 < A$ or $V_3 > B$, conclude $C_2$. As before, $A$ and $B$ respectively are $\chi^2$ values corresponding to left tail and right tail areas of $a/2$ each in the $\chi^2$ distribution with $n - 1$ degrees of freedom. Figure 25.1 shows the action limits for the present example where $n - 1 = 6 - 1 = 5$ and $a = .02$.

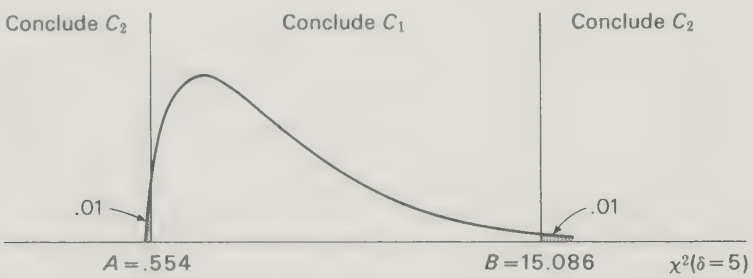



**Figure 25.1**

Thus, for our example, the appropriate decision rule is: If $.554 \leqslant V_3 \leqslant 15.086$, conclude $C_1$; if $V_3 < .554$ or $V_3 > 15.086$, conclude $C_2$.

Since $V_3 = 1.75$, and $.554 \leqslant 1.75 \leqslant 15.086$, the manufacturer should conclude $C_1$, that the population variance is still $\sigma^2 = 2.0$.

**Problem 2**

Refer to Problem 1.

(a) Controlling the risk of making a Type I error at .10, test whether or not $\sigma^2 = 28$. $V_3 =$ _____ ; decision rule: _____ ; conclusion: _____ √

(b) How could the same conclusion reached in (a) have been found from the confidence interval in Problem 1(a)? _____ √

**C. Confidence Interval for the Ratio of Two Population Variances.** Readers who are not familiar with the $F$ distribution should refer first to Topic A, Section 23.

Consider two normal populations with variances $\sigma_1^2$ and $\sigma_2^2$ respectively. Let $s_1^2$ and $s_2^2$ be the sample variances of independent random samples of sizes $n_1$ and $n_2$ from the respective populations. Statistical theory shows that the ratio $s_2^2\,\sigma_1^2/s_1^2\,\sigma_2^2$ has an $F$ distribution with $\delta_2 = n_2 - 1$ and $\delta_1 = n_1 - 1$. The $(1 - a)100$ per cent confidence interval for the ratio $\sigma_2^2/\sigma_1^2$ is given by:

$$\frac{s_2^2}{Ds_1^2} \leqslant \frac{\sigma_2^2}{\sigma_1^2} \leqslant \frac{s_2^2}{Cs_1^2},$$

where $C$ and $D$ denote respectively the $F$ values corresponding to left tail and right tail areas of $a/2$ each in the $F$ distribution with $\delta_2 = n_2 - 1$ and $\delta_1 = n_1 - 1$.

$F$ values corresponding to left tail areas may be obtained from Table A-7, which contains only $F$ values for right tail areas, through the following relationship:

$$\left(\begin{array}{l}F \text{ value corresponding to left tail}\\ \text{area of } a/2 \text{ for } F \text{ distribution}\\ \text{with } \delta_2 = a \text{ and } \delta_1 = b\end{array}\right) = \frac{1}{\left(\begin{array}{l}F \text{ value corresponding to right}\\ \text{tail area of } a/2 \text{ for } F \text{ distribution}\\ \text{with } \delta_2 = b \text{ and } \delta_1 = a\end{array}\right)}.$$

For instance, consider the $F$ distribution with $\delta_2 = 5$ and $\delta_1 = 10$. To find the $F$ value corresponding to a left tail area of .05, we first look up the $F$ value corresponding to a right tail area of .05 for an $F$ distribution with $\delta_2 = 10$ and $\delta_1 = 5$, which is 4.74. We then take the inverse, 1/4.74 = .211, which is the required $F$ value.

**Problem 3**

Find the $F$ value corresponding to a left tail area of .05 for $\delta_2 = 4$ and $\delta_1 = 9$. _____√

Example 3 To compare the uniformity of service life of two types of tires, a random sample of six tires of Type #1 and an independent random sample of eight tires of Type #2 were tested for service life under laboratory conditions. The sample results were $s_1^2 = 4.2$ and $s_2^2 = 17.5$ (thousands of miles squared). We assume both populations are normally distributed.

A 90 per cent confidence interval for the ratio of the variances of the service lives for the two types is desired. We have $a/2 = .05$, $\delta_2 = n_2 - 1 = 8 - 1 = 7$, and $\delta_1 = n_1 - 1 = 6 - 1 = 5$, so that from Table A-7, $C = 1/3.97 = .252$ and $D = 4.88$. The 90 per cent confidence interval is:

$$\frac{17.5}{(4.88)(4.2)} \leqslant \frac{\sigma_2^2}{\sigma_1^2} \leqslant \frac{17.5}{(.252)(4.2)} \quad \text{or} \quad .85 \leqslant \frac{\sigma_2^2}{\sigma_1^2} \leqslant 16.53.$$

We conclude therefore that the ratio $\sigma_2^2/\sigma_1^2$ is between .85 and 16.53 (90 per cent confidence coefficient).

**Problem 4**

A construction firm is comparing actual costs as a per cent of estimated costs for two different types of construction projects. Shown below are the sample results for 10 recent projects of Type #1 and 11 recent projects of Type #2:

| Type #1 | Type #2 |
|---|---|
| $n_1 = 10$ | $n_2 = 11$ |
| $\bar{X}_1 = 109$ | $\bar{X}_2 = 115$ |
| $s_1^2 = 118$ | $s_2^2 = 72$ |

The firm wishes to compare the magnitudes of the population variances of the actual/estimated cost ratios for the two types of projects. Assume the two sets of data represent independent random samples from normal populations.

Answers

2. (a) $s^2 = 14$; $a/2 = .05$; $n - 1 = 12 - 1 = 11$; $A = 4.575$; $B = 19.675$.

The alternative conclusions are: $C_1$: $\sigma^2 = 28$; $C_2$: $\sigma^2 \neq 28$; $V_3 = (11)(14)/28 = 5.50$. The decision rule is: If $4.575 \leqslant V_3 \leqslant 19.675$, conclude $C_1$; if $V_3 < 4.575$ or $V_3 > 19.675$, conclude $C_2$. Since $V_3 = 5.50$ and $4.575 \leqslant 5.50 \leqslant 19.675$, conclude $C_1$.

(b) Since $\sigma^2 = 28$ is contained in the 90 per cent confidence interval, $7.83 \leqslant \sigma^2 \leqslant 33.7$, one would conclude $C_1$ at the .10 level of significance.

3. The $F$ value corresponding to a right tail area of .05 for $\delta_2 = 9$ and $\delta_1 = 4$ is 6.00. Hence the $F$ value corresponding to a left tail of .05 for $\delta_2 = 4$ and $\delta_1 = 9$ is $1/6.00 = .167$.

(a) Construct a 98 per cent confidence interval for the ratio of the population variances $\sigma_2^2/\sigma_1^2$. _____ _____ √

(b) What can you conclude from your confidence interval about how the two population variances compare in magnitude? _____ √

**D. Construction of Decision Rule Concerning Two Population Variances.** We consider the construction of a decision rule for choosing between the following alternative conclusions:

$$C_1: \quad \sigma_1^2 = \sigma_2^2;$$

$$C_2: \quad \sigma_1^2 \neq \sigma_2^2.$$

The construction of decision rules for one-sided alternatives follows in a straightforward manner.

Example 4 Refer to Example 3. Suppose it is desired to test whether or not the service lives for the two types of tires have equal variances. Thus, the alternative conclusions are: $C_1$: $\sigma_1^2 = \sigma_2^2$; $C_2$: $\sigma_1^2 \neq \sigma_2^2$. Suppose the probability of a Type I error is to be controlled at .10.

The appropriate test statistic for the decision rule is $V_4 = s_2^2/s_1^2$. The decision rule for controlling the risk of a Type I error at $a$ has the following form: If $C \leqslant V_4 \leqslant D$, conclude $C_1$; if $V_4 < C$ or $V_4 > D$, conclude $C_2$. The action limits, $C$ and $D$, are obtained from the $F$ distribution with $\delta_2 = n_2 - 1$ and $\delta_1 = n_1 - 1$. $C$ is the $F$ value for a left tail area of $a/2$ and $D$ is the $F$ value for a right tail area of $a/2$. Figure 25.2 shows the action limits for the present example where $a = .10$, $\delta_2 = n_2 - 1 = 8 - 1 = 7$, and $\delta_1 = n_1 - 1 = 6 - 1 = 5$.

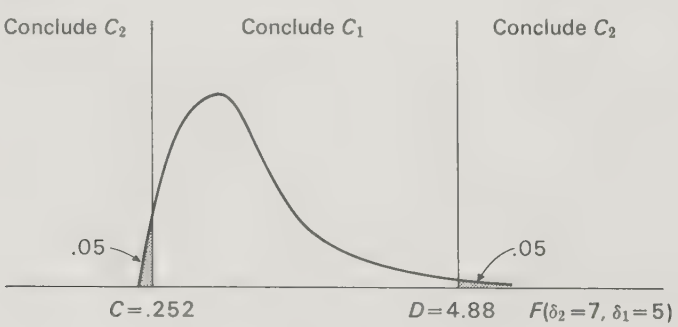



**Figure 25.2**

Thus, for our example, the appropriate decision rule is: If $.252 \leqslant V_4 \leqslant 4.88$, conclude $C_1$; if $V_4 < .252$ or $V_4 > 4.88$, conclude $C_2$.

From Example 3, we have $s_1^2 = 4.2$ and $s_2^2 = 17.5$, so that $V_4 = 17.5/4.2 = 4.17$. Since $.252 \leqslant 4.17 \leqslant 4.88$, we should conclude $C_1$, that the variances of the service lives for the two types of tires are equal.

**Problem 5**

Refer to Problem 4.

(a) Test whether or not the population variances of the actual/estimated cost ratios for the two types of projects are equal. Control the risk of making a Type I error at .02 in constructing the decision rule.

$V_4 =$ _____ ;

decision rule: _____ ;

conclusion: _____ √

(b) How could the same conclusion reached in (a) have been found from the confidence interval constructed in Problem 4? _____ √

## REVIEW PROBLEMS

1. Refer to Problem 4. Construct a 90 per cent confidence interval for the population variance of the actual/estimated cost ratios for construction projects of Type #1. _____ √

2. Refer to Example 3. Past experience with an established brand of tire indicates that $\sigma^2 = 16$ for this brand. Test whether or not the variance of the service life of Type #1 tires is *less* than 16. Assume the probability of concluding that the variance of Type #1 tires is less than 16, when actually it equals 16, is to be held at .10.√

3. A company has two types of machines, eight of Type #1 and three of Type #2. Maintenance costs for each of these machines for the last year have been assembled, and the sample variances for these data are:

| Type #1 | Type #2 |
|---|---|
| $n_1 = 8$ | $n_2 = 3$ |
| $s_1^2 = 40{,}000$ | $s_2^2 = 320{,}000$ |

Assume that these observations are independent random samples from normal populations.

(a) Test whether or not the population variances of maintenance costs per machine are equal for the two types of machines. Control the probability of a Type I error at .10.√

(b) Construct a 90 per cent confidence interval for the ratio of the population variance for Type #2 machines to the population variance for Type #1 machines. _____ √

(c) From your confidence interval in (b), is it a reasonable possibility that the population variance for Type #2 machines is four times as great as the population variance for Type #1 machines? _____

_____

_____ √

Answers

4. (a) $s_1^2 = 118$; $s_2^2 = 72$; $n_1 - 1 = 10 - 1 = 9$; $n_2 - 1 = 11 - 1 = 10$; $a/2 = .01$; $C = 1/4.95 = .202$; $D = 5.26$.
The confidence interval is:

$$\frac{72}{(5.26)\ (118)} \leqslant \frac{\sigma_2^2}{\sigma_1^2} \leqslant \frac{72}{(.202)\ (118)}$$

which gives: $.116 \leqslant \sigma_2^2/\sigma_1^2 \leqslant 3.02$.

**(b) Since, at the 98 per cent confidence level, the confidence interval indicates that the ratio $\sigma_2^2/\sigma_1^2$ may be any value between .116 and 3.02, one has little precise information on the comparative magnitudes of the two variances.**

5. (a) $s_1^2 = 118$; $s_2^2 = 72$; $n_1 - 1 = 10 - 1 = 9$; $n_2 - 1 = 11 - 1 = 10$; $a/2 = .01$; $C = 1/4.95 = .202$; $D = 5.26$.

The alternative conclusions are: $C_1$: $\sigma_1^2 = \sigma_2^2$; $C_2$: $\sigma_1^2 \neq \sigma_2^2$; $V_4 = 72/118 = .610$. The decision rule is: If $.202 \leqslant V_4 \leqslant 5.26$, conclude $C_1$; if $V_4 < .202$ or $V_4 > 5.26$, conclude $C_2$. Since $V_4 = .610$ and $.202 \leqslant .610 \leqslant 5.26$, conclude $C_1$.

(b) Since $\sigma_2^2/\sigma_1^2 = 1$ is contained in the 98 per cent confidence interval, $.116 \leqslant \sigma_2^2/\sigma_1^2 \leqslant 3.02$, one would conclude $C_1$ $(\sigma_1^2 = \sigma_2^2)$ at the .02 level of significance.

## ANSWERS TO REVIEW PROBLEMS

1. $s^2 = 118$; $a/2 = .05$; $n_1 - 1 = 10 - 1 = 9$; $A = 3.325$; $B = 16.919$.

The confidence interval is: $(9)(118)/16.919 \leqslant \sigma^2 \leqslant (9)(118)/3.325$, which gives: $62.77 \leqslant \sigma^2 \leqslant 319.4$.

2. The alternative conclusions are: $C_1$: $\sigma^2 \geqslant 16$; $C_2$: $\sigma^2 < 16$.

The $\chi^2$ value for a lower tail area of .10 and $n - 1 = 6 - 1 = 5$ degrees of freedom is 1.610. The decision rule is: If $V_3 \geqslant 1.610$, conclude $C_1$; if $V_3 < 1.610$, conclude $C_2$.
Here $s^2 = 4.2$. Thus, $V_3 = (5)(4.2)/16 = 1.312$, and since $1.312 < 1.610$, conclude $C_2$.

3. (a) $s_1^2 = 40{,}000$; $s_2^2 = 320{,}000$; $n_1 - 1 = 8 - 1 = 7$; $n_2 - 1 = 3 - 1 = 2$; $a/2 = .05$; $C = 1/19.36 = .0517$; $D = 4.74$.

The alternative conclusions are $C_1$: $\sigma_1^2 = \sigma_2^2$; $C_2$: $\sigma_1^2 \neq \sigma_2^2$.

$V_4 = 320{,}000/40{,}000 = 8.00$. The decision rule is: If $.0517 \leqslant V_4 \leqslant 4.74$, conclude $C_1$; if $V_4 < .0517$ or $V_4 > 4.74$, conclude $C_2$.
Since $V_4 = 8.00 > 4.74$, conclude $C_2$.

**(b) Utilizing the data in (a), the confidence interval is:**

$$\frac{320{,}000}{(4.74)(40{,}000)} \leqslant \frac{\sigma_2^2}{\sigma_1^2} \leqslant \frac{320{,}000}{(.0517)(40{,}000)},$$

**which gives: $1.69 \leqslant \sigma_2^2/\sigma_1^2 \leqslant 155$.**

**(c) Since the 90 per cent confidence interval in (b) does span the value 4, it is possible that the population variance for Type #2 machines is four times as great as the variance for Type #1 machines.**

# 26

# Simple Regression Analysis

In simple regression analysis, we study the statistical relationship between two variables so that one variable can be predicted on the basis of the other.

**A. Simple Regression Model.** Consider two variables, $X$ and $Y$, where $X$ is the "independent" variable which takes on specified values, and $Y$ is the "dependent" variable which is related to $X$ according to the following statistical model:

$$Y_i = \alpha + \beta X_i + \epsilon_i .$$

Here $\alpha$ and $\beta$ are unknown parameters (constants) and $X_i$ is a specified value of $X$. The component $\epsilon_i$ is a random variable, representing a residual error amount, which is assumed to be normally distributed with a mean of zero and an unknown standard deviation $\sigma_{Y.X}$. It is assumed that $\sigma_{Y.X}$ is the same for all specified values of $X$, and that the residual error amounts, $\epsilon_i$, are statistically independent.

Let $\mu_{Y.X}$ be the mean of the distribution of $Y$ for a specified value of $X$. It is seen from the above statement of the statistical model that, for a given value of $X$, $Y$ is a normally distributed variable with mean $\mu_{Y.X} = \alpha + \beta X$ and standard deviation $\sigma_{Y.X}$.

From a sample of pairs of $X$ and $Y$ values, estimators of the unknown parameters $\alpha$ and $\beta$ can be obtained by the method of *least squares.* The formulas for these estimators and also for the estimator of $\sigma_{Y.X}$ are given in Table 26.1.

**Table 26.1**

| Parameter | Estimator | Formula of Estimator |
|---|---|---|
| $\beta$ | $b$ | $b = \dfrac{\Sigma(Y-\overline{Y})(X-\overline{X})}{\Sigma(X-\overline{X})^2}$ |
| $\alpha$ | $a$ | $a = \overline{Y} - b\overline{X}$ |
| $\sigma_{Y.X}$ | $s_{Y.X}$ | $s_{Y.X} = \sqrt{\dfrac{\Sigma(Y-a-bX)^2}{n-2}}$ |

Here $n$ is the number of sample pairs of $X$ and $Y$ values, and $\overline{X}$ and $\overline{Y}$ are the sample means of the $X$ and $Y$ values respectively.

More convenient computational formulas for $b$ and $s_{Y.X}$ are:

$$b = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{n\Sigma X^2 - (\Sigma X)(\Sigma X)},$$

$$s_{Y.X} = \sqrt{\frac{Q_E}{n-2}}, \quad \text{where} \quad Q_E = \Sigma Y^2 - a\Sigma Y - b\Sigma XY.$$

Both $a$ and $b$ should be computed to several digits in order to obtain a numerically accurate value for $Q_E$.

Example 1 A firm has a fleet of six automobiles used by field personnel. It wishes to study the relationship between the fuel cost in cents per mile ($Y$) of operating an automobile and the automobile's age in years ($X$). Data collected for the six automobiles during a representative period are given in Table 26.2. The firm assumes that there is a linear relationship between the mean fuel cost per mile ($\mu_{Y.X}$) and age ($X$), and wishes to estimate the regression equation $\mu_{Y.X} = \alpha + \beta X$.

**Table 26.2**

| Age $X$ | Cost $Y$ | $X^2$ | $XY$ | $Y^2$ |
|---|---|---|---|---|
| 1 | 2.0 | 1 | 2.0 | 4.00 |
| 3 | 3.0 | 9 | 9.0 | 9.00 |
| 4 | 2.5 | 16 | 10.0 | 6.25 |
| 2 | 2.0 | 4 | ____ | ____ |
| 1 | 2.0 | ____ | ____ | ____ |
| 7 | 3.5 | ____ | 24.5 | ____ |
| Total 18 | ____ | ____ | ____ | ____ |

$\bar{X} = 18/6 = 3$; $\bar{Y} =$_____ $/6 =$_____ .

**Problem 1**

Complete the calculations in Table 26.2.√

Example 2 Using the data in Table 26.2 and the formulas for $a$ and $b$ presented earlier, we obtain:

$$b = \frac{6(51.5) - (18)(15.0)}{6(80) - (18)(18)} = .2500;$$

$$a = 2.50 - .2500(3) = 1.75.$$

The estimated regression equation therefore is:

$$\bar{Y}_X = 1.75 + .2500X,$$

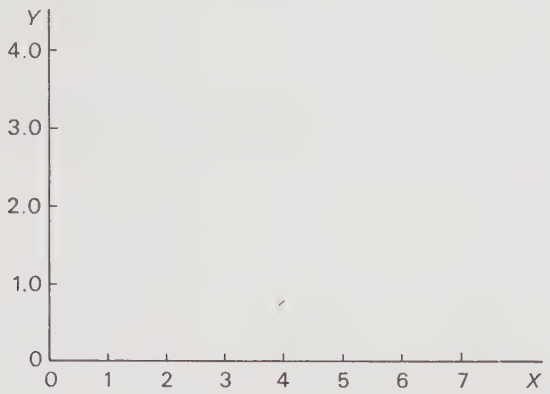
where $\bar{Y}_X$ is the point estimator of $\mu_{Y.X}$, the mean of the distribution of $Y$ for a specified value of $X$. For $X = 2$, we estimate the mean fuel cost per mile to be:

$$\bar{Y}_X = 1.75 + .2500(2) = 2.25.$$

**Problem 2**

Refer to Example 2.
(a) Calculate $\bar{Y}_X$ for $X = 3$. _____ √

(b) Plot the estimated regression equation on a graph, as well as the six pairs of $X$ and $Y$ values.




(c) Are $X$ and $Y$ positively or negatively related? _____ √

**Problem 3**

Refer to Example 1. The firm wishes to estimate the standard deviation $\sigma_{Y.X}$.
(a) Calculate $s_{Y.X}$. _____
(b) What are the units of $s_{Y.X}$? _____ √

**B. Confidence Interval for $\mu_{Y.X}$.** Frequently it is desired to construct a confidence interval for the mean of the distribution of $Y$ for a specified value $X^*$, that is, for $\mu_{Y.X^*} = \alpha + \beta X^*$. The desired confidence interval is:

$$\overline{Y}_{X^*} - t\, s_{\overline{Y}_{X^*}} \leqslant \mu_{Y.X^*} \leqslant \overline{Y}_{X^*} + t s_{\overline{Y}_{X^*}}.$$

Here $\overline{Y}_{X^*}$ is the point estimator of $\mu_{Y.X^*}$ when $X = X^*$, namely $\overline{Y}_{X^*} = a + bX^*$. The symbol $s_{\overline{Y}_{X^*}}$ denotes the sample estimator of the standard deviation of the sampling distribution of $\overline{Y}_{X^*}$:

$$s_{\overline{Y}_{X^*}} = s_{Y.X}\sqrt{\frac{1}{n} + \frac{n(X^* - \overline{X})^2}{n\Sigma X^2 - (\Sigma X)(\Sigma X)}}.$$

Finally, the $t$ value in the confidence interval is obtained from the $t$ distribution with $n - 2$ degrees of freedom, corresponding to the specified confidence coefficient. (Note that if $n$ is large, $t$ may be replaced by $z$ in the confidence interval.)

Example 3 Refer to Example 1. Suppose the firm wants to construct a 95 per cent confidence interval for the mean fuel cost per mile of operating a two-year-old automobile.

In this case, $X^* = 2$, $\overline{Y}_{X^*} = 2.25$ (see Example 2), $s_{Y.X} = .306$ (see Problem 3(a)), and:

$$s_{\overline{Y}_{X^*}} = .306\sqrt{\frac{1}{6} + \frac{6(2-3)^2}{6(80) - (18)(18)}} = .139.$$

For a 95 per cent confidence coefficient and $n - 2 = 6 - 2 = 4$ degrees of freedom, $t = 2.776$. Hence, the confidence interval is:

$$2.25 - 2.776(.139) \leqslant \mu_{Y.X^*} \leqslant 2.25 + 2.776(.139) \quad \text{or} \quad 1.86 \leqslant \mu_{Y.X^*} \leqslant 2.64.$$

Thus we estimate that the mean fuel cost of operating two-year-old automobiles is between 1.86 and 2.64 cents per mile (95 per cent confidence coefficient).

**Answers**

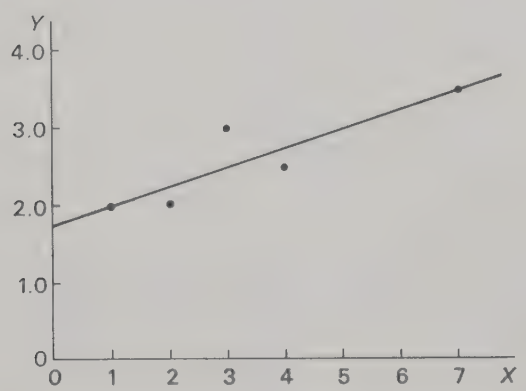
1.

| $X$ | $Y$ | $X^2$ | $XY$ | $Y^2$ |
|---|---|---|---|---|
| 2 | 2.0 | 4 | 4.0 | 4.00 |
| 1 | 2.0 | 1 | 2.0 | 4.00 |
| 7 | 3.5 | 49 | 24.5 | 12.25 |
| 18 | 15.0 | 80 | 51.5 | 39.50 |

$\overline{Y} = 15.0/6 = 2.50.$

2. (a) For $X = 3$, $\overline{Y}_X = 1.75 + .2500(3) = 2.50.$

(b)




(c) The positive slope of the regression line ($b > 0$) suggests that $X$ and $Y$ are positively related.

3. (a) $Q_E = 39.50 - 1.75(15.0) - .2500(51.5) = .3750;$

$s_{Y.X} = \sqrt{.3750/(6-2)} = .306.$

(b) The units of $s_{Y.X}$ are cents per mile.

**Problem 4**

Refer to Example 3.

(a) Construct a 95 per cent confidence interval for the mean fuel cost per mile of operating a three-year-old automobile. (*Hint*: $\overline{Y}_{X^*}$ was calculated in Problem 2.) _____ √

(b) Is the width of the interval in (a) the same as that of the interval obtained in Example 3? _____

(c) Why does the width of the confidence interval vary? _____

_____

_____ √

**C. Prediction Interval for $Y^*$.** Frequently, it is desired to construct a prediction interval for a new observation $Y$ which will be obtained for a specified value of $X^*$, that is, to construct a prediction interval for $Y^* = \alpha + \beta X^* + \epsilon$. The desired prediction interval is:

$$\overline{Y}_{X^*} - ts_{Y^*} \leqslant Y^* \leqslant \overline{Y}_{X^*} + ts_{Y^*}.$$

As before, $\overline{Y}_{X*}$ denotes the point estimator of $\mu_{Y.X*}$, that is, $\overline{Y}_{X*} = a + bX^*$. The symbol $s_{Y*}$ denotes the sample standard deviation:

$$s_{Y*} = \sqrt{s^2_{Y.X} + s^2_{\overline{Y}_{X*}}}.$$

Thus, the standard deviation $s_{Y*}$ in the prediction interval for a new observation is made up of two components: (1) $s^2_{Y.X}$, the inherent variation of $Y$ itself, and (2) $s^2_{\overline{Y}_{X*}}$, the sampling error in using $\overline{Y}_{X*}$ as an estimator of the true mean $\mu_{Y.X*}$. The $t$ value in the prediction interval is obtained from the $t$ distribution with $n - 2$ degrees of freedom, corresponding to the specified confidence coefficient.

Example 4 Refer to Example 1. The firm wishes to construct a 95 per cent prediction interval for the fuel cost per mile of operating a two-year-old automobile which it intends to purchase.

In this case, a prediction interval for $Y^*$ is desired when $X^* = 2$. We have $\overline{Y}_{X*} = 2.25$ (see Example 2), $s_{Y.X} = .306$ (see Problem 3), $s_{\overline{Y}_{X*}} = .139$ (see Example 3), and:

$$s_{Y*} = \sqrt{(.306)^2 + (.139)^2} = .336.$$

For a 95 per cent confidence coefficient and $n - 2 = 6 - 2 = 4$ degrees of freedom, $t = 2.776$. Hence, the prediction interval is:

$$2.25 - 2.776(.336) \leqslant Y^* \leqslant 2.25 + 2.776(.336) \quad \text{or} \quad 1.32 \leqslant Y^* \leqslant 3.18.$$

Thus, we expect the fuel cost of operating the two-year-old car to be between 1.32 and 3.18 cents per mile (95 per cent confidence coefficient).

**Problem 5**

(a) Why is the prediction interval in Example 4 wider than the confidence interval for $\mu_{Y.X}$ for two-year-old cars in Example 3? _____

_____

(b) Would it be appropriate to set up a prediction interval for a fifteen-year-old car in Example 1? _____ Discuss. _____

_____

_____ √

**Problem 6**

Refer to Example 4. Suppose the firm is anticipating the purchase of a three-year-old automobile for the fleet.

(a) Construct a 90 per cent prediction interval for the fuel cost per mile of operating a three-year-old car. (*Hint*: $\overline{Y}_{X*}$ and $s_{\overline{Y}_{X*}}$ are found in Problem 4.) _____ √

(b) Consider the three-year-old car presently owned by the firm. Does the fuel cost of operating that car fall within the prediction interval obtained in (a)? _____ √

**D. Confidence Interval for $\beta$.** The confidence interval for $\beta$, the slope of the regression line, is:

$$b - ts_b \leqslant \beta \leqslant b + ts_b,$$

where $b$ is the point estimator of $\beta$ and $s_b$ is the point estimator of $\sigma_b$, the standard deviation of the sampling distribution of $b$:

$$s_b = s_{Y.X}\sqrt{\frac{n}{n\Sigma X^2 - (\Sigma X)(\Sigma X)}}.$$

**Answers**

4. (a) Here $X^* = 3$, $\bar{Y}_{X^*} = 2.50$(Problem 2), $s_{Y.X} = .306$(Problem 3), and

$$s_{\bar{Y}_{X^*}} = .306\sqrt{\frac{1}{6} + \frac{6(3-3)^2}{6(80)-(18)(18)}} = .125.$$

For $n - 2 = 6 - 2 = 4$ degrees of freedom and a 95 per cent confidence coefficient, $t = 2.776$. Hence, the confidence interval is: $2.50 - 2.776(.125) \leqslant \mu_{Y.X^*} \leqslant 2.50 + 2.776(.125)$ or $2.15 \leqslant \mu_{Y.X^*} \leqslant 2.85$.

(b) The width of the confidence interval in (a) is $2.85 - 2.15 = .70$ while the confidence interval in Example 3 has a width of $2.64 - 1.86 = .78$. Hence the interval in (a) is narrower than the interval in Example 3.

(c) The width of the confidence interval varies because the term $[n(X^* - \bar{X})^2]/[n\Sigma X^2 - (\Sigma X)(\Sigma X)]$ in the formula for $s_{\bar{Y}_{X^*}}$ increases in value as $|X^* - \bar{X}|$ becomes larger.

5. (a) The prediction interval is wider because $s_{Y^*}$ contains the term $s^2_{Y.X}$ in addition to $s^2_{\bar{Y}_{X^*}}$, that is, $s_{Y^*} = \sqrt{s^2_{Y.X} + s^2_{\bar{Y}_{X^*}}}$.

(b) Since 15 is far outside the range of past experience, there are hazards in extrapolating the linear regression relation for predicting the mean fuel cost per mile for fifteen-year-old cars.

6. (a) Here $X^* = 3$, $\bar{Y}_{X^*} = 2.50$(Problem 4), $s_{\bar{Y}_{X^*}} = .125$(Problem 4), $s_{Y.X} = .306$(Problem 3).

Hence: $$s_{Y^*} = \sqrt{(.306)^2 + (.125)^2} = .331.$$

For a 90 per cent confidence coefficient and $n - 2 = 6 - 2 = 4$ degrees of freedom, $t = 2.132$. Hence, the prediction interval is: $2.50 - 2.132(.331) \leqslant Y^* \leqslant 2.50 + 2.132(.331)$ or $1.79 \leqslant Y^* \leqslant 3.21$.

(b) Yes, for the three-year-old car presently owned $Y = 3.0$.

The $t$ value in the confidence interval is obtained from the $t$ distribution with $n - 2$ degrees of freedom, corresponding to the specified confidence coefficient.

Example 5 Refer to Example 1. For this example, $b = .2500$ and $s_{Y.X} = .306$. Thus:

$$s_b = .306\sqrt{\frac{6}{6(80)-(18)(18)}} = .060.$$

If a 95 per cent confidence interval is desired, the appropriate $t$ value is 2.776 (from the $t$ distribution with $n - 2 = 6 - 2 = 4$ degrees of freedom). The confidence interval is:

$$.2500 - 2.776(.060) \leqslant \beta \leqslant .2500 + 2.776(.060) \quad \text{or} \quad .08 \leqslant \beta \leqslant .42.$$

**Problem 7**

The slope $\beta$ indicates how much the mean fuel cost per mile changes with an increase of one year in the age of the automobile. Does the confidence interval for $\beta$ obtained in Example 5 support the maxim that older automobiles cost more per mile to operate? _____ Explain. _____ √

**E. Sample Coefficient of Determination and Test for $\beta = 0$.** In regression analysis, the total variation in the $Y$ values (the total sum of squares) can be partitioned into two components; first, the variation associated with the

relationship of $Y$ and $X$, and second, the variation which is residual after the relationship of $Y$ and $X$ has been taken into consideration. This partitioning of the total sum of squares may be denoted: $Q = Q_R + Q_E$, where:

$$Q = \text{total sum of squares} = \Sigma(Y - \overline{Y})^2,$$

$$Q_R = \text{regression sum of squares} = \Sigma(\overline{Y}_X - \overline{Y})^2,$$

$$Q_E = \text{residual sum of squares} = \Sigma(Y - \overline{Y}_X)^2 = \Sigma(Y - a - bX)^2.$$

A computational formula for $Q_E$ has already been given, namely:

$$Q_E = \Sigma Y^2 - a\Sigma Y - b\Sigma XY.$$

A computational formula for $Q$ is:

$$Q = \frac{n\Sigma Y^2 - (\Sigma Y)(\Sigma Y)}{n}.$$

Finally, $Q_R$ is obtained from the relation:

$$Q_R = Q - Q_E.$$

Example 6 The *sample coefficient of determination* is $r^2 = Q_R/Q$. It indicates the proportion of the total variation in $Y$ associated with the regression of $Y$ on $X$.

**Problem 8**

For the regression problem in Example 1, $Q = [6(39.50) - (15.0)(15.0)]/6 = 2.0000$ and $Q_E = .3750$ (see Problem 3). Calculate: $Q_R$ = _____ ; $r^2$ = _____ √

**Problem 9**

(a) What is the range of values that $r^2$ may take on? (*Hint*: $Q = Q_R + Q_E$.) _____ √

(b) Is knowledge of $X$ of any help in predicting $Y$ if $r^2$ is approximately zero? _____ If $r^2$ is approximately one? _____ √

Example 7 An analysis of variance table can be used to test the alternative conclusions: $C_1$: $\beta = 0$, $C_2$: $\beta \neq 0$, as shown in Table 26.3.

**Table 26.3**

| Source | *SS* | *df* | *MS* | *E(MS)* |
|---|---|---|---|---|
| Regression | $Q_R$ | 1 | $MS_R = \frac{Q_R}{1}$ | $\sigma^2_{Y.X} + \beta^2\Sigma(X - \overline{X})^2$ |
| Residual | $Q_E$ | $n - 2$ | $MS_E = \frac{Q_E}{n-2} = s^2_{Y.X}$ | $\sigma^2_{Y.X}$ |
| Total | $Q$ | $n - 1$ | | |

The appropriate test statistic is $V_R = MS_R/MS_E$. If $C_1$ is true, $V_R$ follows the $F$ distribution with $\delta_2 = 1$ and $\delta_1 = n - 2$. The appropriate decision rule for controlling the Type I error is: If $V_R \leq A$, conclude $C_1$; if $V_R > A$, conclude $C_2$. Here $A$ is the $F$ value from the $F$ distribution with $\delta_2 = 1$ and $\delta_1 = n - 2$, corresponding to the specified level of significance (risk of a Type I error) for the test.

**Answers**

7. Yes, since the confidence interval $.08 \leqslant \beta \leqslant .42$ includes only positive values of $\beta$.

8. $Q_R = 2.0000 - .3750 = 1.6250$;
   $r^2 = 1.6250/2.0000 = .8125$.

9. (a) Since $Q = Q_R + Q_E$, $Q_R \geqslant 0$, and $Q_E \geqslant 0$, it follows that $0 \leqslant r^2 \leqslant 1$.

   (b) If $r^2$ is approximately zero, very little of the variation in $Y$ is associated with the regression on $X$, and knowledge of $X$ is of little help in predicting $Y$. On the other hand, if $r^2$ is approximately one, most of the variation in $Y$ is associated with the regression on $X$, and knowledge of $X$ is of help in predicting $Y$ (though it may not be sufficient to provide useful results).

**Problem 10**

Refer to Example 1. Shown in Table 26.4 is the partially completed analysis of variance table for this example.

**Table 26.4**

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 1.6250 | 1 | 1.6250 |
| Residual | _____ | ___ | _____ |
| Total | 2.0000 | ___ | |

(a) Complete Table 26.4. (*Hint*: $Q_E$ is given in Problem 8.)
(b) State the alternative conclusions for testing whether or not the regression coefficient (slope of the regression line) $\beta$ is zero. _____√
(c) Calculate: $V_R$ = _____
(d) What is the appropriate action limit for the decision rule, given that the probability of making a Type I error is to be controlled at .05? $A$ = _____
(e) What conclusion about $\beta$ should be accepted? _____√
(f) Is the result of this test consistent with the 95 per cent confidence interval for $\beta$ obtained in Example 5?
_____
(g) Why is there interest in testing whether the regression coefficient $\beta$ is zero? _____
_____√

## REVIEW PROBLEMS

1. A university administrator studied the relationship between the cost of operating an academic department and the total student-hours of teaching and supervision undertaken in the department, for eleven departments for the most recent academic year. The results are summarized below in a convenient form:

$X$ = number of student-hours (in thousands),
$Y$ = cost ($ thousands),
$n = 11$ $\quad n\Sigma X^2 - (\Sigma X)(\Sigma X) = 16{,}810{,}000$,
$\Sigma X = 7040$ $\quad n\Sigma Y^2 - (\Sigma Y)(\Sigma Y) = 8{,}910{,}000$,
$\Sigma Y = 4235$ $\quad n\Sigma XY - (\Sigma X)(\Sigma Y) = 9{,}245{,}500$.

(a) Assuming simple linear regression is appropriate, estimate the regression equation. _____

(b) Construct a 90 per cent confidence interval for the mean cost $\mu_{Y.X}$ of operating departments with 681 thousand student-hours. (*Hint*: $Q_E = 347{,}725$.) _____

(c) Construct a 90 per cent prediction interval for the cost of operating a specific department with 681 thousand student-hours. _____

(d) Assume the regression model is appropriate for the coming academic year. The Department of Engineering will have 681 thousand student-hours during the coming year. Would the confidence interval in (b) or the prediction interval in (c) be appropriate for estimating the cost of operating the Department of Engineering during the coming year? _____

(e) Construct a 90 per cent confidence interval for $\beta$. _____

(f) From your confidence interval in (e), would you conclude that $\beta$ is positive? _____

(g) Calculate the sample coefficient of determination. _____

(h) Construct an analysis of variance table and test whether or not $\beta = 0$. Control the risk of a Type I error at .05.

Answers

10. (a)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 1.6250 | 1 | 1.6250 |
| Residual | .3750 | 4 | .0938 |
| Total | 2.0000 | 5 | |

(b) $C_1: \beta = 0$; $C_2: \beta \neq 0$.

(c) $V_R = 1.6250/.0938 = 17.3$.

(d) $A = F(\delta_2 = 1, \delta_1 = 4)$ for a right tail area of .05 = 7.71.

(e) Since $V_R > A$, that is, $17.3 > 7.71$, conclude $C_2$.

(f) **Yes. Note that the 95 per cent confidence interval in Example 5 does not span zero.**

(g) Interest in testing whether the regression coefficient $\beta$ is zero exists because when $\beta = 0$ there is no statistical relationship between $X$ and $Y$.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $b = 9{,}245{,}500/16{,}810{,}000 = .55$;
$a = (4235/11) - .55(7040/11) = 385 - .55(640) = 33$.
Hence, the estimated regression equation is: $\overline{Y}_X = 33 + .55X$.

(b) Here $X^* = 681$, $s_{Y.X} = \sqrt{347{,}725/(11-2)} = 197$,

$$\overline{Y}_{X^*} = 33 + .55(681) = 408, \text{ and}$$

$$s_{\overline{Y}_{X^*}} = 197\sqrt{\frac{1}{11} + \frac{11(681-640)^2}{16{,}810{,}000}} = 197\sqrt{.0920} = 60.$$

For $n - 2 = 11 - 2 = 9$ degrees of freedom and a 90 per cent confidence coefficient, $t = 1.833$. Hence, the confidence interval is:

$$408 - 1.833(60) \leqslant \mu_{Y.X^*} \leqslant 408 + 1.833(60) \quad \text{or} \quad 298 \leqslant \mu_{Y.X^*} \leqslant 518.$$

(c) Here $X^* = 681$, $s_{Y.X} = 197$, $\overline{Y}_{X^*} = 408$, $s_{\overline{Y}_{X^*}} = 60$ (see answer to (b)), and $s_{Y^*} = \sqrt{(197)^2 + (60)^2} = 206$.
For $n - 2 = 11 - 2 = 9$ degrees of freedom and a 90 per cent confidence coefficient, $t = 1.833$. Hence, the prediction interval is:

$$408 - 1.833(206) \leqslant Y^* \leqslant 408 + 1.833(206) \quad \text{or} \quad 30 \leqslant Y^* \leqslant 786.$$

(d) Since a prediction of a specific value of $Y$ is desired, the prediction interval in (c) is appropriate.

(e) $b = .55$ and $s_b = 197\sqrt{11/16{,}810{,}000} = .159$.
For $n - 2 = 11 - 2 = 9$ degrees of freedom and a 90 per cent confidence coefficient, $t = 1.833$. Hence, the confidence interval for $\beta$ is:

$$.55 - 1.833(.159) \leqslant \beta \leqslant .55 + 1.833(.159) \quad \text{or} \quad .26 \leqslant \beta \leqslant .84.$$

(f) **Yes, because the confidence interval $.26 \leqslant \beta \leqslant .84$ contains only positive values of $\beta$.**

## ANSWERS TO REVIEW PROBLEMS

**1. cont'd.**

(g) $Q = 8{,}910{,}000/11 = 810{,}000$ and $Q_E = 347{,}725$, thus
$Q_R = 810{,}000 - 347{,}725 = 462{,}275$ and $r^2 = 462{,}275/810{,}000 = .57$.

(h)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 462,275 | 1 | 462,275 |
| Residual | 347,725 | 9 | 38,636 |
| Total | 810,000 | 10 | |

With a level of significance of .05, the appropriate action limit is $A = F(\delta_2 = 1, \delta_1 = 9)$ for a right tail area of .05 = 5.12. Hence, the decision rule is: If $V_R \leqslant 5.12$, conclude $C_1$ ($\beta = 0$); if $V_R > 5.12$, conclude $C_2$ ($\beta \neq 0$). Here $V_R = 462{,}275/38{,}636 = 11.96$. Since $V_R > A$, that is, $11.96 > 5.12$, we conclude $C_2$.

# 27

# Multiple and Curvilinear Regression Analysis

In multiple regression analysis, two or more independent variables are used in predicting the dependent variable of interest. In curvilinear regression analysis, there may be only one independent variable, but it is used in several different forms. For the sake of brevity, the term "multiple regression analysis" generally will be used to describe both multiple and curvilinear regression. In this section, we consider the case of two predicting variables (or two forms of one predicting variable).

**A. Multiple Regression Model.** The statistical model underlying multiple regression analysis, with two predicting variables, has the form:

$$Y_i = \alpha + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i.$$

Here $Y_i$ is an observed value of the dependent variable of interest, $X_{i1}$ and $X_{i2}$ denote specified values of the two predicting variables, $X_1$ and $X_2$, and $\alpha$, $\beta_1$, and $\beta_2$ are unknown parameters (constants). The component $\epsilon_i$ is a random variable, representing a residual error amount, which is assumed to be normally distributed with a mean of zero and an unknown standard deviation $\sigma_{Y.X_1X_2}$. It is assumed that $\sigma_{Y.X_1X_2}$ is the same for all pairs of values of $X_1$ and $X_2$. The residual error amounts, $\epsilon_i$, are assumed to be statistically independent.

Let $\mu_{Y.X_1X_2}$ be the mean of the distribution of $Y$ for a specified pair of values of $X_1$ and $X_2$. From the above statement of the statistical model, it is seen that, for given values of $X_1$ and $X_2$, $Y$ is a normally distributed random variable with a mean equal to $\mu_{Y.X_1X_2} = \alpha + \beta_1 X_1 + \beta_2 X_2$ and a standard deviation of $\sigma_{Y.X_1X_2}$.

**Example 1** In Example 1 of Section 26, we described the case of a firm with a fleet of six automobiles used by field personnel that wished to study the relationship between the fuel cost per mile of operating an automobile and its age in years.

Suppose the firm believes that the fuel cost per mile ($Y$) is related not only to the age ($X$) of the automobile, but also to the age squared ($X^2$), as follows: $Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \epsilon$, where $X_1 = X$ and $X_2 = X^2$. This special case of multiple regression analysis is called curvilinear regression, because the functional relationship between the mean fuel cost per mile and age is a curve, that is, $\mu_{Y.X_1X_2} = \alpha + \beta_1 X_1 + \beta_2 X_2 = \alpha + \beta_1 X + \beta_2 X^2$.

**Problem 1**

A firm has eight marketing regions and wishes to predict the sales in each region on the basis of the population of the region and a measure of the extent by which average per capita disposable income in the region deviates from the average per capita disposable income for the regions. The multiple regression model to be employed is: $Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \epsilon$. Here $Y$ denotes annual sales (in thousands of units), $X_1$ denotes population (in millions), and $X_2$ denotes a measure of the deviation of per capita disposable income from the regional average (in $ thousands).

(a) What does $\beta_1$ represent in this model? _____

_____

_____

(b) What are the units of $\epsilon$? _____ √

**Answers**

1. (a) $\beta_1$ represents the change in the mean annual sales in a region, in thousands of units, when the population of the region is increased by one million persons and $X_2$ is held constant.
   (b) Thousands of units.

**B. Point Estimators of Regression Parameters.** From a sample of triplets of $Y$, $X_1$, and $X_2$ values, estimates of the unknown regression parameters, $\alpha$, $\beta_1$, and $\beta_2$, can be obtained, using the method of least squares. The desired estimators for $\alpha$, $\beta_1$, and $\beta_2$ may be obtained by solving simultaneously the following equations, called *normal equations,* for the estimators $a$, $b_1$, and $b_2$, respectively.

$$\begin{aligned}\Sigma Y &= na + b_1\Sigma X_1 + b_2\Sigma X_2,\\ \Sigma X_1 Y &= a\Sigma X_1 + b_1\Sigma X_1^2 + b_2\Sigma X_1 X_2,\\ \Sigma X_2 Y &= a\Sigma X_2 + b_1\Sigma X_1 X_2 + b_2\Sigma X_2^2.\end{aligned}$$

The estimator for $\sigma_{Y.X_1X_2}$ is:

$$s_{Y.X_1X_2} = \sqrt{\frac{Q_E}{n-3}}, \quad \text{where } Q_E = \Sigma Y^2 - a\Sigma Y - b_1\Sigma X_1 Y - b_2\Sigma X_2 Y.$$

Here $n$ is the number of sample triplets of $Y$, $X_1$, and $X_2$ values. The estimates $a$, $b_1$, and $b_2$ should be computed to several digits in order to obtain a numerically accurate value for $Q_E$.

(In the discussion of simple regression analysis, in Section 26, the formulas for the estimators $a$ and $b$ are obtained by solving a set of normal equations similar to those given above.)

Example 2 Refer to Example 1. Data on fuel cost per mile and age for the firm's six automobiles are given in Table 27.1.

**Table 27.1**

| Age $X_1 = X$ | Age Squared $X_2 = X^2$ | Cost $Y$ | $X_1^2$ | $X_2^2$ | $Y^2$ | $X_1X_2$ | $X_1Y$ | $X_2Y$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2.0 | 1 | 1 | 4.00 | 1 | 2.0 | 2.0 |
| 3 | 9 | 3.0 | 9 | 81 | 9.00 | 27 | 9.0 | 27.0 |
| 4 | 16 | 2.5 | 16 | 256 | 6.25 | 64 | 10.0 | 40.0 |
| 2 | 4 | 2.0 | 4 | 16 | 4.00 | 8 | 4.0 | 8.0 |
| 1 | 1 | 2.0 | 1 | 1 | 4.00 | 1 | 2.0 | 2.0 |
| 7 | 49 | 3.5 | 49 | 2401 | 12.25 | 343 | 24.5 | 171.5 |
| 18 | 80 | 15.0 | 80 | 2756 | 39.50 | 444 | 51.5 | 250.5 |

To estimate the regression parameters $\alpha$, $\beta_1$, and $\beta_2$, we substitute the above results into the normal equations to obtain the following three equations in $a$, $b_1$, and $b_2$:

$$\begin{aligned}15.0 &= 6a + 18b_1 + 80b_2,\\ 51.5 &= 18a + 80b_1 + 444b_2,\\ 250.5 &= 80a + 444b_1 + 2756b_2.\end{aligned}$$

Solving the above three equations simultaneously (calculations not shown), we obtain: $a = 1.69249$, $b_1 = .294220$ and $b_2 = -.00563584$. Therefore, the estimated regression equation is:

$$\bar{Y}_{X_1X_2} = 1.69249 + .294220X_1 - .00563584X_2.$$

Here $\overline{Y}_{X_1X_2}$ is a point estimator of $\mu_{Y.X_1X_2}$, the mean of the distribution of $Y$ for specified values of $X_1$ and $X_2$. In the $X$ notation, the estimated regression equation is:

$$\overline{Y}_{X_1X_2} = 1.69249 + .294220X - .00563584X^2.$$

The estimate of $\sigma_{Y.X_1X_2}$ is obtained by substituting the above results into the formula for $s_{Y.X_1X_2}$. Substitution yields:

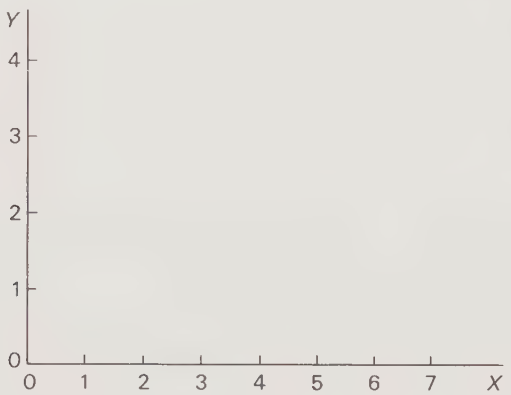
$$Q_E = 39.50 - 1.69249(15.0) - .294220(51.5) + .00563584(250.5) = .3721$$

and

$$s_{Y.X_1X_2} = \sqrt{\frac{.3721}{6-3}} = .352.$$

**Problem 2**

Refer to Example 2. Plot the estimated regression equation on a graph, as well as the six pairs of $X$ and $Y$ values.√




**Problem 3**

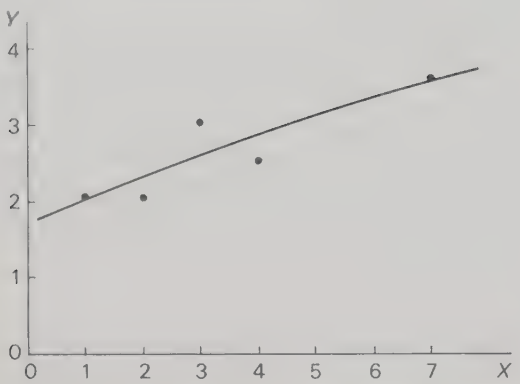
Refer to Problem 1. The sales, population, and disposable income deviation data for the eight marketing regions are given in Table 27.2.

**Table 27.2**

| Population $X_1$ | Disposable Income Deviation $X_2$ | Sales $Y$ | $X_1^2$ | $X_2^2$ | $Y^2$ | $X_1X_2$ | $X_1Y$ | $X_2Y$ |
|---|---|---|---|---|---|---|---|---|
| 3 | −.6 | 1 | 9 | .36 | 1 | −1.8 | 3 | − .6 |
| 1 | +.6 | 8 | 1 | .36 | 64 | .6 | 8 | 4.8 |
| 5 | −.2 | 8 | 25 | .04 | 64 | −1.0 | 40 | −1.6 |
| 4 | +.2 | 6 | 16 | .04 | 36 | .8 | 24 | 1.2 |
| 3 | −.5 | 1 | 9 | .25 | 1 | −1.5 | 3 | − .5 |
| 3 | +.1 | 6 | 9 | .01 | 36 | .3 | 18 | .6 |
| 5 | +.2 | 12 | — | — | — | — | — | — |
| 8 | +.2 | 18 | — | — | — | — | — | — |
| 32 | 0 | 60 | 158 | 1.14 | 670 | 0 | 300 | 9.9 |

**Answers**

**2.**




(a) Fill in the blanks in Table 27.2.
(b) Estimate the parameters $\alpha, \beta_1$, and $\beta_2$. _____ _____ _____
(c) State the estimated regression equation. _____√
(d) Estimate $\sigma_{Y.X_1X_2}$. _____√

**C. Point Estimator of $\mu_{Y.X_1X_2}$.** A point estimator of the mean $\mu_{Y.X_1X_2}$ of the distribution of $Y$ for specified values of $X_1$ and $X_2$ is given by $\overline{Y}_{X_1X_2}$, where $\overline{Y}_{X_1X_2} = a + b_1X_1 + b_2X_2$.

Example 3 Refer to Example 1 and Example 2, where the estimated regression equation relating fuel cost per mile and age is $\overline{Y}_{X_1X_2} = 1.69249 + .294220X - .00563584X^2$. A point estimate of the mean fuel cost per mile of operating, for instance, a three-year-old car is obtained by substituting $X = 3$ into the regression equation. The substitution yields:

$$\overline{Y}_{X_1X_2} = 1.69249 + .294220(3) - .00563584(3)^2 = 2.52 \text{ cents per mile.}$$

**Problem 4**

Refer to Example 3. Estimate the mean fuel cost per mile of operating seven-year-old cars. _____√

**Problem 5**

Refer to Problem 3. The estimated regression equation is $\overline{Y}_{X_1X_2} = -.5000 + 2.0000X_1 + 8.6842X_2$. Estimate the mean sales when the population is 4 million and the deviation of per capita disposable income is +$.3 thousand. _____√

**D. Sample Coefficient of Multiple Determination and Test for $\beta_1 = \beta_2 = 0$.** As in simple regression analysis, the total sum of squares for the $Y$ values ($Q$) can be partitioned into two components: (1) $Q_R$, the sum of squares associated with the regression of $Y$ on $X_1$ and $X_2$; and (2) $Q_E$, the sum of squares which is residual after the regression of $Y$ on $X_1$ and $X_2$ is taken into account. These sums of squares are as follows:

$$Q = \Sigma(Y - \overline{Y})^2, \quad Q_R = \Sigma(\overline{Y}_{X_1X_2} - \overline{Y})^2, \quad \text{and} \quad Q_E = \Sigma(Y - \overline{Y}_{X_1X_2})^2.$$

The computational formulas are:

$$Q = \frac{n\Sigma Y^2 - (\Sigma Y)(\Sigma Y)}{n},$$
$$Q_E = \Sigma Y^2 - a\Sigma Y - b_1\Sigma X_1Y - b_2\Sigma X_2Y,$$
$$Q_R = Q - Q_E.$$

Example 4 The *sample coefficient of multiple determination* is $R^2 = Q_R/Q$. It indicates the proportion of the total variation in $Y$ associated with the regression of $Y$ on $X_1$ and $X_2$.

**Problem 6**

Refer to Example 2. For the cost per mile and age data given in the example, $Q = 2.0000$ and $Q_E = .3721$. Find: $Q_R =$ _____ and $R^2 =$ _____ √

**Problem 7**

(a) What is the range of values $R^2$ may take on? _____ √

(b) Is knowledge of $X_1$ and $X_2$ of any help in predicting $Y$ if $R^2$ is approximately zero? _____

_____

_____ √

Example 5 An analysis of variance table can be used to test the alternative conclusions: $C_1$: $\beta_1 = \beta_2 = 0$; $C_2$: not all $\beta_j = 0$, as shown in Table 27.3.

**Table 27.3**

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | $Q_R$ | 2 | $MS_R = Q_R/2$ |
| Residual | $Q_E$ | $n-3$ | $MS_E = Q_E/(n-3) = s^2_{Y.X_1X_2}$ |
| Total | $Q$ | $n-1$ | |

The appropriate test statistic is $V_R = MS_R/MS_E$. If $C_1$ is true, $V_R$ follows the $F$ distribution with $\delta_2 = 2$ and $\delta_1 = n - 3$. The decision rule appropriate for controlling the Type I error is: If $V_R \leq A$, conclude $C_1$; if $V_R > A$, conclude $C_2$, where $A$ is the $F$ value from the $F$ distribution with $\delta_2 = 2$ and $\delta_1 = n - 3$, corresponding to the specified level of significance (risk of a Type I error) for the test.

**Problem 8**

Refer to Example 2, involving the regression of cost per mile on age and age squared. Table 27.4 is the partially completed analysis of variance table.

**Table 27.4**

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 1.6279 | 2 | .8140 |
| Residual | .3721 | ___ | ____ |
| Total | 2.0000 | ___ | |

(a) Complete Table 27.4.

(b) State the alternatives for testing whether or not both $\beta_1$ and $\beta_2$ are zero. _____ _____

(c) Calculate: $V_R =$ _____ √

(d) The appropriate action limit for the decision rule, if the risk of a Type I error is to be controlled at .10, is $F(\delta_2 = 2, \delta_1 = 3)$ for a right tail area of .10, which is 5.46. Which conclusion about $\beta_1$ and $\beta_2$ should be accepted? _____

(e) Does conclusion $C_2$ imply that neither $\beta_1$ nor $\beta_2$ is zero? _____

(f) What is the significance if both $\beta_1$ and $\beta_2$ are zero? _____

_____

_____ √

Answers

3. (a)

| $X_1$ | $X_2$ | $Y$ | $X_1^2$ | $X_2^2$ | $Y^2$ | $X_1X_2$ | $X_1Y$ | $X_2Y$ |
|---|---|---|---|---|---|---|---|---|
| 5 | +.2 | 12 | 25 | .04 | 144 | 1.0 | 60 | 2.4 |
| 8 | +.2 | 18 | 64 | .04 | 324 | 1.6 | 144 | 3.6 |

(b) To estimate the parameters $\alpha$, $\beta_1$, and $\beta_2$, we set up the normal equations as shown below and solve for $a$, $b_1$, and $b_2$, respectively.

$$\begin{aligned} 60 &= 8a + 32b_1 + 0b_2, \\ 300 &= 32a + 158b_1 + 0b_2, \\ 9.9 &= 0a + 0b_1 + 1.14b_2. \end{aligned}$$

From the last equation, we obtain $b_2 = 9.9/1.14 = 8.6842$. Solving the first two equations simultaneously, we obtain $a = -.5000$ and $b_1 = 2.0000$.

(c) $\overline{Y}_{X_1X_2} = -.5000 + 2.0000X_1 + 8.6842X_2$.

(d) To estimate $\sigma_{Y.X_1X_2}$, we substitute into the formula for $Q_E$ to obtain:

$$Q_E = 670 + .5000(60) - 2.0000(300) - 8.6842(9.9) = 14.0264.$$

Hence, $s_{Y.X_1X_2} = \sqrt{14.0264/(8-3)} = 1.6748$.

4. $X = 7$. Thus, $\overline{Y}_{X_1X_2} = 1.69249 + .294220(7) - .00563584(7)^2 = 3.48$ cents per mile.

5. $X_1 = 4$; $X_2 = .3$. Thus, $\overline{Y}_{X_1X_2} = -.5000 + 2.0000(4) + 8.6842(.3) = 10.105$ or 10,105 units of product.

6. $Q_R = 2.0000 - .3721 = 1.6279$ and $R^2 = 1.6279/2.0000 = .814$.

7. (a) Since $Q = Q_R + Q_E$, $Q_R \geqslant 0$, and $Q_E \geqslant 0$, it follows that $0 \leqslant R^2 \leqslant 1$.
(b) If $R^2$ is approximately zero, very little of the variation in $Y$ is associated with the regression on $X_1$ and $X_2$, and knowledge of $X_1$ and $X_2$ is of little help in predicting $Y$.

8. (a)

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 1.6279 | 2 | .8140 |
| Residual | .3721 | 3 | .1240 |
| Total | 2.0000 | 5 | |

(b) $C_1$: $\beta_1 = \beta_2 = 0$; $C_2$: not all $\beta_j = 0$.
(c) $V_R = .8140/.1240 = 6.56$.
(d) Since $V_R > A$, that is, $6.56 > 5.46$, conclude $C_2$.
(e) Not necessarily; conclusion $C_2$ implies that either $\beta_1$ is not zero, that $\beta_2$ is not zero, or that both are not equal to zero.
(f) If both $\beta_1$ and $\beta_2$ are zero, then there is no relationship between $X_1$ and $X_2$ on the one hand and the dependent variable $Y$ on the other hand.

**Problem 9**

Refer to Problem 3 involving the regression of sales on population and deviation of disposable income. For this problem:

(a) Set up the analysis of variance table for regression. (*Hint*: $Q_E = 14.0264$; see Problem 3.) √

(b) Calculate the sample coefficient of multiple determination. _____

(c) State the alternatives for testing whether or not $\beta_1 = \beta_2 = 0$, and conduct the test. Control the risk of a Type I error at .05.√

## REVIEW PROBLEMS

1. A financial analyst is studying the relationship between current price of a company's common stock and current dividends per share and rate of growth of dividends during the previous five years. The analyst assumes that the relationship between the variables can be described by the linear multiple regression model. The results for a sample of 100 companies are:

| | | |
|---|---|---|
| $\Sigma X_1 = 240$ | $\Sigma X_1^2 = 800$ | $\Sigma X_1 X_2 = 1000$ |
| $\Sigma X_2 = 360$ | $\Sigma X_2^2 = 1500$ | $\Sigma X_1 Y = 17{,}680$ |
| $\Sigma Y = 6400$ | $\Sigma Y^2 = 500{,}000$ | $\Sigma X_2 Y = 26{,}100$ |

where $Y$ = current price (\$), $X_1$ = current dividends (\$), $X_2$ = rate of growth of dividends (in per cent).

(a) Estimate the regression equation in this problem. _____ √

(b) Estimate the mean current price of common stock when the current dividend is \$2.00 per share and the rate of growth of dividends is 4 per cent. _____

(c) Set up the analysis of variance table for regression.√

(d) Calculate the sample coefficient of multiple determination. _____

(e) Test whether or not $\beta_1 = \beta_2 = 0$. Control the risk of a Type I error at .05. (Note: $F(\delta_2 = 2, \delta_1 = 97)$ for a right tail area of .05 = 3.10.) √

Answers

**9. (a)**

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 205.9736 | 2 | 102.9868 |
| Residual | 14.0264 | 5 | 2.8053 |
| Total | 220. | 7 | |

where $Q = [8(670) - (60)(60)]/8 = 220$
and $Q_R = Q - Q_E = 220 - 14.0264 = 205.9736$.

**(b)** $R^2 = 205.9736/220 = .936$.

**(c)** $C_1: \beta_1 = \beta_2 = 0$; $C_2$: not all $\beta_j = 0$.
$A = F(\delta_2 = 2, \delta_1 = 5)$ for a right tail area of .05 = 5.79; $V_R = 102.9868/2.8053 = 36.7$.
Since $V_R > A$, that is, $36.7 > 5.79$, conclude $C_2$.

## ANSWERS TO REVIEW PROBLEMS

**1. (a) The normal equations are:**

(1) $6400 = 100a + 240b_1 + 360b_2$,
(2) $17{,}680 = 240a + 800b_1 + 1000b_2$,
(3) $26{,}100 = 360a + 1000b_1 + 1500b_2$.

An easy way of solving these equations is to multiply equation (2) by 1.5, and then subtract it from equation (3). Both $a$ and $b_2$ drop out, and we obtain $b_1 = 2.1$. Then we replace $b_1$ by 2.1 in equations (1) and (2) and solve these simultaneously. We then find $a = 10$ and $b_2 = 13.6$. The estimated regression equation is:

$$\overline{Y}_{X_1X_2} = 10 + 2.1X_1 + 13.6X_2.$$

**(b)** $X_1 = 2.00$; $X_2 = 4$. Thus: $\overline{Y}_{X_1X_2} = 10 + 2.1(2.00) + 13.6(4) = \$68.60$.

**(c)**

| Source | *SS* | *df* | *MS* |
|---|---|---|---|
| Regression | 46,488 | 2 | 23,244 |
| Residual | 43,912 | 97 | 453 |
| Total | 90,400 | 99 | |

**(d)** $R^2 = 46{,}488/90{,}400 = .514$.

**(e)** $V_R = 23{,}244/453 = 51.3$ and $A = 3.10$. Since $V_R > A$, that is, $51.3 > 3.10$, conclude that not all $\beta_j = 0$.

# 28

# Correlation Analysis

In this section we consider the degree of association or correlation between two or more random variables which are jointly normally distributed.

**A. Simple Correlation Coefficient.** We assume that the two random variables under consideration, $X$ and $Y$, have a bivariate normal distribution. This bivariate distribution has the property that the conditional distribution of $Y$, given a specified value of $X$, is normal, with mean $\mu_{Y \cdot X} = \alpha + \beta X$ and standard deviation $\sigma_{Y \cdot X}$. (Note that this property is identical to the simple regression model discussed in Section 26.) In addition, a bivariate normal distribution has the property that the marginal distribution of $Y$ is normal with mean $\mu_Y$ and standard deviation $\sigma_Y$. Further, the conditional distribution of $X$, given $Y$, is normal with mean $\mu_{X \cdot Y} = \alpha' + \beta' Y$ and standard deviation $\sigma_{X \cdot Y}$ and the marginal distribution of $X$ is normal with mean $\mu_X$ and standard deviation $\sigma_X$.

A parameter of the bivariate normal distribution (which is used frequently to describe the degree of association between $X$ and $Y$) is the *correlation coefficient*. It is denoted by $\rho$ (Greek rho) and is defined by $\rho = \text{Cov}(X, Y)/\sigma_X \sigma_Y$ where $\text{Cov}(X, Y)$ is the population covariance of $X$ and $Y$. The square of $\rho$, $\rho^2$, is called the *coefficient of determination*. (The correlation coefficient also is discussed in Section 6, Topic F.)

An estimator of $\rho$ based on a sample of $n$ pairs of $X$ and $Y$ values is called the *sample correlation coefficient*. It is denoted by $r$ where:

$$r = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{[n\Sigma X^2 - (\Sigma X)(\Sigma X)]\,[n\Sigma Y^2 - (\Sigma Y)(\Sigma Y)]}}.$$

An estimator of $\rho^2$ is $r^2$ which, in terms of the notation of Section 26, Topic E, can be expressed simply: $r^2 = Q_R/Q$.

Example 1 A personnel officer is studying performances of job applicants on two tests given when the applicant contacts the firm. The first test measures mental ability; the second measures potential for success in the job.

The test-score results of a sample of eleven applicants are shown in Table 28.1 on page 240.

The personnel officer wishes to assess the degree of association between the two test scores, and calculates the sample correlation coefficient:

$$r = \frac{11(17{,}599) - (448)(418)}{\sqrt{[11(19{,}270) - (448)(448)]\,[11(17{,}800) - (418)(418)]}} = \frac{6325}{\sqrt{(11{,}266)(21{,}076)}} = +.41.$$

A positive value for $r$ indicates a positive correlation between $X$ and $Y$, that is, high mental ability scores tend to be associated with high success potential scores, and conversely, low mental ability scores tend to be associated with low success potential scores.

Table 28.1

| Applicant | Mental Ability $X$ | Potential $Y$ | $X^2$ | $XY$ | $Y^2$ |
|---|---|---|---|---|---|
| A | 37 | 63 | 1369 | 2331 | 3969 |
| B | 40 | 42 | 1600 | 1680 | 1764 |
| C | 36 | 41 | 1296 | 1476 | 1681 |
| D | 49 | 39 | 2401 | 1911 | 1521 |
| E | 36 | 38 | 1296 | 1368 | 1444 |
| F | 40 | 49 | 1600 | 1960 | 2401 |
| G | 39 | 25 | 1521 | 975 | 625 |
| H | 47 | 29 | 2209 | 1363 | 841 |
| I | 32 | 15 | 1024 | 480 | 225 |
| J | 65 | 52 | 4225 | 3380 | 2704 |
| K | 27 | 25 | 729 | 675 | 625 |
| Total | 448 | 418 | 19,270 | 17,599 | 17,800 |

**Problem 1**

(a) What range of values may be taken on by $r$? (*Hint*: We noted earlier that $0 \leqslant r^2 \leqslant 1$.) _____
By $\rho$? _____ √

(b) Refer to Example 1. If $r$ had been negative rather than positive, what would be the nature of the association of $X$ and $Y$? _____

(c) What is the essential distinction between $r = +1$ and $r = -1$? _____ √

(d) What information about a causal relationship between $X$ and $Y$ does $\rho$ furnish? _____ √

**Problem 2**

In a study of the relationship between family income and amount of a firm's product consumed by the family, a sample of fifteen families yielded the following results:

$X$ = family income (in $ thousands)
$Y$ = product consumption (in units)

$n = 15$ $\quad \Sigma X^2 = 600$
$\Sigma X = 80$ $\quad \Sigma XY = 400$
$\Sigma Y = 60$ $\quad \Sigma Y^2 = 300$

(a) Calculate the sample correlation coefficient. $r =$ _____

(b) Does the sign of the correlation coefficient suggest that high income families tend to consume more of the product than low income families? _____ √

**B. Test for $\rho = 0$.** A nonzero value of $\rho$ indicates that a statistical relationship exists between $X$ and $Y$. Hence, frequently we are interested in testing the alternative conclusions: $C_1 : \rho = 0$; $C_2 : \rho \neq 0$. Since $\rho = 0$ if and only if $\beta = 0$ ($\beta$ is the slope of the regression line), we can test whether or not $\rho = 0$ through testing $C_1 : \beta = 0$, $C_2 : \beta \neq 0$, the test for which was discussed in Section 26, Topic E. Alternatively, we can test directly whether or not $\rho = 0$ by means of the test statistic $V = r\sqrt{n-2} \,/\, \sqrt{1-r^2}$. When $C_1$ is true, $V$ follows a $t$ distribution with $n - 2$ degrees of freedom. Hence, the appropriate decision rule to control the Type I error at $\alpha$ is: If $A_1 \leqslant V \leqslant A_2$, conclude $C_1$; if $V < A_1$ or $V > A_2$, conclude $C_2$. Here $A_1$ and $A_2$ are $t$ values corresponding to tail areas each of $\alpha/2$ in the $t$ distribution with $n - 2$ degrees of freedom.

Example 2 In the situation of Example 1, to test whether or not $\rho = 0$, we calculate $V = (.41)\sqrt{11-2}\,/\sqrt{1-(.41)^2} = 1.35$. Using a level of significance of $\alpha = .05$, we find that $A_1 = -2.262$ and $A_2 = +2.262$ (these are $t$ values corresponding to a total area in both tails of .05 for the $t$ distribution with $n - 2 = 11 - 2 = 9$ degrees of freedom). Since $-2.262 \leqslant 1.35 \leqslant +2.262$, we conclude $C_1(\rho = 0)$.

**Problem 3**

Refer to Problem 2. Test whether or not family income and product consumption are correlated. Control the risk of a Type I error at .02. √

**C. Confidence Interval for $\rho$.** To construct a confidence interval for $\rho$, we use Fisher's transformation, which has the form $w = 1.1513 \log_{10}[(1+r)/(1-r)]$. For large sample sizes, $w$ is approximately normally distributed with mean $\mu_w = 1.1513 \log_{10}[(1+\rho)/(1-\rho)]$ and standard deviation $\sigma_w = 1/\sqrt{n-3}$. Samples of more than ten usually are considered sufficiently large for the normal approximation to be applicable.

First we construct a confidence interval for $\mu_w$:

$$w - z\sigma_w \leqslant \mu_w \leqslant w + z\sigma_w,$$

where $z$ is the standard normal deviate which corresponds to the specified confidence coefficient. Then we derive the confidence interval for $\rho$ from the confidence interval for $\mu_w$, as shown in the following example.

Example 3 In a sample of 110 employees, a personnel officer found the sample correlation coefficient between length of employment and punctuality to be $r = +.34$. To construct a 95 per cent confidence interval for $\rho$, we first calculate $w$ and $\sigma_w$. Table A-8 provides values of $w$ for given values of $r$ where $w = 1.1513 \log_{10}[(1+r)/(1-r)]$. Note that $w$ and $r$ always have the same sign.

Consulting Table A-8, we find $w = +.3541$ when $r = +.34$. Also, $\sigma_w = 1/\sqrt{110-3} = .0967$. The $z$ value corresponding to a 95 per cent confidence interval is 1.96. Hence, the confidence interval for $\mu_w$ is:

$$.3541 - 1.96(.0967) \leqslant \mu_w \leqslant .3541 + 1.96(.0967) \quad \text{or} \quad .1646 \leqslant \mu_w \leqslant .5436.$$

Since $\mu_w = 1.1513 \log_{10}[(1+\rho)/(1-\rho)]$, the confidence limits for $\rho$ are found in Table A-8 corresponding to the $w$ values, .1646 and .5436, the confidence limits for $\mu_w$. From Table A-8, we obtain .16 and .50 respectively. Hence, the 95 per cent confidence interval for $\rho$ is: $+.16 \leqslant \rho \leqslant +.50$.

**Problem 4**

Refer to Problem 2.

(a) Construct a 99 per cent confidence interval for $\rho$. _____ √

(b) Note that the confidence limits in (a) are not symmetrical around $r = +.784$. What does this indicate about the shape of the sampling distribution of $r$? _____

_____

(c) Suppose you wished to test: $C_1: \rho = +.5$, $C_2: \rho \neq +.5$, controlling the risk of a Type I error at .01. Use your confidence interval in (a) to determine the appropriate conclusion for the test. _____ √

**D. Multiple Correlation Coefficient.** Multiple correlation analysis is an extension of simple correlation analysis, in that more than two random variables are under consideration. For the sake of compactness, the discussion here will be limited to three variables, denoted $X_1$, $X_2$, and $Y$. These are assumed to have a multivariate normal distribution. For such a distribution, all conditional distributions are normal. For instance the conditional distribution of $Y$, given specified values of $X_1$ and $X_2$, is normal with mean $\mu_{Y.X_1X_2} = \alpha + \beta_1 X_1 + \beta_2 X_2$ and standard deviation

Answers

1. (a) Both $r$ and $\rho$ may take on any value between $-1$ and $+1$ inclusive.
   (b) A negative value for $r$ would indicate that low mental ability scores tend to be associated with high success potential scores, and vice versa.
   (c) **Both cases imply that all points fall on the regression line, but in one case the slope is positive and in the other it is negative.**
   (d) **Correlation analysis provides no direct information about causal relationships.**

2. (a) $r = \dfrac{(15)(400) - (80)(60)}{\sqrt{[15(600) - (80)(80)][15(300) - (60)(60)]}} = +.784.$

   (b) Yes, since $r$ is positive.

3. $C_1: \rho = 0$; $C_2: \rho \neq 0$.
   Since $r = .784$ (from Problem 2(a)), $V = .784\sqrt{15-2}/\sqrt{1-(.784)^2} = 4.55$. Since $\alpha = .02$ and $n - 2 = 15 - 2 = 13$, the action limits are $A_1 = -2.650$ and $A_2 = +2.650$. Since $V > A_2$, that is, $4.55 > 2.650$, conclude $C_2$.

4. (a) Interpolating in Table A-8, we find $w = 1.055$ corresponds to $r = .784$ (from Problem 2(a)). Since $n = 15$, $\sigma_W = 1/\sqrt{15-3} = .289$. The $z$ value corresponding to a 99 per cent confidence interval is 2.58. Hence the confidence interval for $\mu_W$ is: $1.055 - 2.58(.289) \leq \mu_W \leq 1.055 + 2.58(.289)$ which gives $.309 \leq \mu_W \leq 1.801$. From Table A-8, we find that $r = .30$ corresponds to $w = .309$ and $r = .95$ corresponds to $w = 1.801$. Thus, the 99 per cent confidence interval for $\rho$ is: $+.30 \leq \rho \leq +.95$.
   (b) The lack of symmetry of the confidence limits indicates that the sampling distribution of $r$ is skewed, when $\rho \neq 0$.
   (c) Since $\rho = +.5$ falls in the 99 per cent confidence interval, $+.30 \leq \rho \leq +.95$, we conclude $C_1$.

$\sigma_{Y.X_1X_2}$. (Note that this property is identical to the multiple regression model discussed in Section 27.) In addition, the marginal distributions of $X_1$, $X_2$, and $Y$ are normal. For instance, the marginal distribution of $Y$ is normal with mean $\mu_Y$ and standard deviation $\sigma_Y$.

A parameter which is used frequently to describe the degree of association between $Y$ and the $X$ variables is the *multiple correlation coefficient*. It is denoted by $\rho_m$ and is defined:

$$\rho_m = +\sqrt{1 - \frac{\sigma^2_{Y.X_1X_2}}{\sigma^2_Y}}.$$

The range of values that $\rho_m$ may take on is from zero to one. The square of $\rho_m$, $\rho_m^2$, is called the *coefficient of multiple determination*.

An estimator of $\rho_m^2$, in terms of the notation of Section 27, Topic D, is $R^2 = 1 - (Q_E/Q) = Q_R/Q$ where $R^2$ is the *sample coefficient of multiple determination*. The positive square root of $R^2$, $R = +\sqrt{Q_R/Q}$, is called the *sample multiple correlation coefficient* and is an estimator of $\rho_m$, the multiple correlation coefficient.

Example 4 In Section 27, Review Problem 1, the sample coefficient of multiple determination between current stock prices and current dividends per share and rate of growth of dividends during the previous five years was $R^2 = .514$. Hence, the sample multiple correlation coefficient is $R = \sqrt{.514} = .72$.

**E. Test for $\rho_m = 0$.** A positive value of $\rho_m$ indicates that a statistical relationship exists between $Y$ and $X_1$ and $X_2$. Hence, frequently we are interested in testing the alternative conclusions: $C_1: \rho_m = 0$, $C_2: \rho_m > 0$. Since $\rho_m = 0$ if and only if $\beta_1 = \beta_2 = 0$, we can test whether or not $\rho_m = 0$ through the test for $C_1: \beta_1 = \beta_2 = 0$, $C_2$: not all $\beta_j = 0$, the test for which was discussed in Section 27, Topic D.

**F. Partial Correlation Coefficients.** Partial correlation coefficients are parameters that measure the degree of association between two random variables after each of them has been corrected for the effect of other variables. In the discussion here, only the case where two random variables are corrected for the effect of a third random variable will be considered.

Let $X_1, X_2$, and $Y$ denote the three random variables under consideration. Let $\rho_{12}$ be the simple correlation coefficient for the variables $X_1$ and $X_2$, $\rho_{1Y}$ the correlation coefficient for the variables $X_1$ and $Y$, and so on. The partial correlation coefficient for the variables $X_1$ and $X_2$, corrected for $Y$, is denoted $\rho_{12.Y}$ and is defined as follows:

$$\rho_{12.Y} = (\rho_{12} - \rho_{1Y}\rho_{2Y})/\sqrt{(1 - \rho_{1Y}^2)(1 - \rho_{2Y}^2)}.$$

Analogously, the partial correlation coefficient for the variables $X_2$ and $Y$, corrected for $X_1$, is denoted $\rho_{2Y.1}$ and is:

$$\rho_{2Y.1} = (\rho_{2Y} - \rho_{12}\rho_{1Y})/\sqrt{(1 - \rho_{12}^2)(1 - \rho_{1Y}^2)},$$

and so on. Partial correlation coefficients can take on values between $-1$ and $+1$ inclusive.

The sample estimator of a partial correlation coefficient is obtained by substituting the sample correlation coefficients for the population correlation coefficients in the above formulas. For instance, the sample estimator of $\rho_{12.Y}$ is:

$$r_{12.Y} = (r_{12} - r_{1Y}r_{2Y})/\sqrt{(1 - r_{1Y}^2)(1 - r_{2Y}^2)}.$$

Example 5 A company is studying the relationship between sales volume ($Y$), number of calls by salesmen on prospective customers ($X_1$), and level of advertising expenditures ($X_2$) for fifteen marketing regions. For a given period, the following results were obtained: $r_{12} = .40$, $r_{1Y} = .61$, and $r_{2Y} = .65$.

Suppose the company wishes to measure the correlation between sales volume and the level of advertising expenditures after adjusting both of these variables for the influence of the number of calls on prospective customers. Thus, a sample estimator of $\rho_{2Y.1}$ is to be calculated:

$$r_{2Y.1} = [.65 - (.40)(.61)]/\sqrt{(1 - .40^2)(1 - .61^2)} = .56.$$

Observe that $r_{2Y.1} = .56$ and $r_{2Y} = .65$. This indicates that the strength of association between sales volume and the level of advertising expenditures is of the same order of magnitude whether or not both variables are adjusted for the effect of the number of calls on prospective customers.

**Problem 5**

Refer to Example 5.

(a) Calculate $r_{12.Y}$ and interpret its meaning. $r_{12.Y} =$ _____

_____

_____

(b) Compare $r_{12}$ and $r_{12.Y}$. What does the difference in the magnitudes of these two variables imply?

_____

_____ √

**G. Spearman Rank Correlation Coefficient.** The Spearman rank correlation coefficient, or more briefly, the rank correlation coefficient, is a measure of the agreement between two variables expressed in ranks.

Example 6 Two personnel officers were asked to rank five job applicants from the best to the worst in terms of their suitability for a position. The results of the ranking are displayed in Table 28.2.

Answers

5. (a) $r_{12.Y} = [.40 - (.61)(.65)] / \sqrt{(1 - .61^2)(1 - .65^2)} = .006$.
$r_{12.Y}$ is a sample estimate of the partial correlation coefficient for the number of calls on prospective customers and the level of advertising expenditures, both adjusted for the effect of the sales volume variable.

(b) $r_{12.Y} = .006$, $r_{12} = .40$. The comparison shows that the apparent correlation between the number of calls on prospective customers and the level of advertising expenditures as indicated by the simple correlation coefficient ($r_{12} = .40$) results from their common correlation with the sales volume variable. When the effect of the variation in sales volume is removed from each variable, it is seen that the two variables are nearly uncorrelated ($r_{12.Y} = .006$).

Table 28.2

| | Ranking Provided by Officer #1 | Officer #2 | | |
|---|---|---|---|---|
| Applicant | $X$ | $Y$ | $(X - Y)$ | $(X - Y)^2$ |
| A | 4 | 3 | +1 | 1 |
| B | 1 (best) | 2 | −1 | 1 |
| C | 3 | 5 | −2 | 4 |
| D | 2 | 1 (best) | +1 | 1 |
| E | 5 | 4 | +1 | 1 |
| Total | | | 0 | 8 |

The rank correlation coefficient is computed using the following formula:

$$r = 1 - [6\Sigma(X - Y)^2 / n(n^2 - 1)].$$

Note that the sum of the values in the $(X - Y)$ column should equal zero. This provides a check on your arithmetic. Here $\Sigma(X - Y)^2 = 8$ and $n = 5$, thus $r = 1 - [6(8)/5(5^2 - 1)] = .60$.

The rank correlation coefficient may take on values between −1 and +1, and is interpreted in a manner analogous to the simple correlation coefficient. For instance, $r = .60$, in the example, indicates a positive relation between the rankings of the two officers.

Since the rank correlation coefficient can be computed quickly, it is sometimes used where the regular correlation coefficient could be employed with more cumbersome calculations.

**Problem 6**

Refer to Example 6.

(a) Does the value of the rank correlation coefficient depend on whether the best applicant is ranked 1 or 5 (and the others are ranked accordingly)?_____ Why? _____

(b) What is required for $\Sigma(X - Y)^2$ to equal zero?_____ What is $r$ in that case? _____

(c) For the situation in Example 6, contrast the following two possible results: $r = +1$, $r = -1$.

_____ √

(d) Suppose, in Example 6, the results of the ranking were those displayed in Table 28.3. Calculate the rank correlation coefficient. $r =$ _____

(e) Why must $\Sigma(X - Y) = 0$ always?_____ √

**Table 28.3**

| Applicant | $X$ | $Y$ |
|---|---|---|
| A | 4 | 3 |
| B | 1 | 2 |
| C | 3 | 4 |
| D | 2 | 1 |
| E | 5 | 5 |

## REVIEW PROBLEMS

1. A stock market analyst is studying the statistical relationship between stock market price movements for three stocks, #1, #2, and #3. For each stock, data have been collected on 24 successive monthly price changes. Some of the data are summarized below. Here $X_i$ represents a monthly price change for stock $i$.

$$n\Sigma X_1^2 - (\Sigma X_1)^2 = 96, \qquad n\Sigma X_2^2 - (\Sigma X_2)^2 = 24,$$
$$n\Sigma X_1 X_2 - (\Sigma X_1)(\Sigma X_2) = -8, \qquad r_{13} = 0, \qquad r_{23} = +.72.$$

The analyst is willing to assume that successive sets of stock price changes are statistically independent and are jointly normally distributed.

(a) Estimate the simple correlation coefficient for stocks #1 and #2. $r_{12} =$ _____

(b) Test whether or not $\rho_{12} = 0$. Control the risk of a Type I error at .02. √

(c) Construct a 95 per cent confidence interval for $\rho_{12}$. _____

_____

(d) Decide whether or not $\rho_{12} = -.4$. Control the risk of a Type I error at .05. (*Hint*: Use your confidence interval in (c).) _____ √

(e) Estimate the partial correlation coefficient for price movements in stocks #1 and #3, corrected for price movements in stock #2. _____

(f) Interpret the partial correlation coefficient obtained in (e). _____

_____

_____ √

(g) Consider a stock portfolio consisting of two stocks. Will the total market value of the portfolio be more stable if the two stocks have negatively or positively correlated price changes? _____ √

2. Refer to Table 28.1. Rank the scores on each of the two tests, giving rank 1 to the highest score in each test. For scores which are tied, give each a rank equal to the average of the ranks which otherwise would be assigned to those scores. For instance, in the mental ability tests, there are two scores of 40 which would otherwise have the ranks 4 and 5. In this case, give each of the scores a rank of 4.5. Calculate the rank correlation coefficient. $r =$ _____ √

Answers

6. (a) No, because $(X - Y)^2$ remains invariant and $\Sigma(X - Y)$ remains equal to zero.
   (b) The rankings of $X$ and $Y$ must be identical, that is, $X = Y$ for all applicants. From the formula for $r$ we see that $r = +1$ when $\Sigma(X - Y)^2 = 0$.
   (c) If $r = +1$, the two officers have provided identical rankings for the applicants. If $r = -1$, the two rankings are completely opposite.
   (d) $n = 5$; $\Sigma(X - Y)^2 = 4$; $r = 1 - [6(4)/5(5^2 - 1)] = .80$.
   (e) $\Sigma(X - Y) = 0$ because $\Sigma X = \Sigma Y$ always.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $r_{12} = -8/\sqrt{(96)\,(24)} = -.17$.
   (b) $C_1 : \rho_{12} = 0$; $C_2 : \rho_{12} \neq 0$.
   Since $r = -.17$, $V = -.17\sqrt{24 - 2}\,/\sqrt{1 - (-.17)^2} = -.81$.
   Since $\alpha = .02$ and $n - 2 = 24 - 2 = 22$, the action limits are $A_1 = -2.508$ and $A_2 = +2.508$. Since $A_1 \leqslant V \leqslant A_2$, that is, $-2.508 \leqslant -.81 \leqslant +2.508$, conclude $C_1$.
   (c) Since $n = 24$ and $r = -.17$, $\sigma_W = 1/\sqrt{24 - 3} = .218$, and from Table A-8, $w = -.1717$. For a 95 per cent confidence coefficient, $z = 1.96$. Hence the confidence interval for $\mu_W$ is: $-.1717 - 1.96(.218) \leqslant \mu_W \leqslant -.1717 + 1.96(.218)$, which gives $-.599 \leqslant \mu_W \leqslant +.256$. From Table A-8, we find the confidence limits of $\rho$ to be $-.54$ and $+.25$. Thus, the 95 per cent confidence interval for $\rho$ is: $-.54 \leqslant \rho \leqslant +.25$.
   (d) $C_1 : \rho_{12} = -.4$; $C_2 : \rho_{12} \neq -.4$.
   Since $\rho_{12} = -.4$ falls in the interval $-.54 \leqslant \rho \leqslant +.25$, we conclude $C_1$.
   (e) $r_{13.2} = [0 - (-.17)(.72)]\,/\sqrt{[1 - (-.17)^2]\,[1 - (.72)^2]} = .18$.
   (f) $r_{13.2} = .18$ indicates that when the effects of price changes in stock #2 have been removed from the price changes for stocks #1 and #3, the price changes for stocks #1 and #3 may be slightly positively correlated. Note that when the price changes in stock #2 are ignored, stocks #1 and #3 exhibit zero correlation.
   (g) **Since negatively correlated price changes imply that price changes for the two stocks tend to be in opposite directions, while positively correlated price changes imply that price changes for the two stocks tend to be in the same direction, a given portfolio of the two stocks will have a more stable total market value if the price changes are negatively correlated.**

2.

| Applicant | Rank, Mental Ability Test $X$ | Rank, Test of Potential $Y$ | $(X - Y)$ | $(X - Y)^2$ |
|---|---|---|---|---|
| A | 7 | 1 | 6 | 36 |
| B | 4.5 | 4 | .5 | .25 |
| C | 8.5 | 5 | 3.5 | 12.25 |
| D | 2 | 6 | −4 | 16 |
| E | 8.5 | 7 | 1.5 | 2.25 |
| F | 4.5 | 3 | 1.5 | 2.25 |
| G | 6 | 9.5 | −3.5 | 12.25 |
| H | 3 | 8 | −5 | 25 |
| I | 10 | 11 | −1 | 1 |
| J | 1 | 2 | −1 | 1 |
| K | 11 | 9.5 | 1.5 | 2.25 |
| Total | | | 0 | 110.5 |

Here $\Sigma(X - Y)^2 = 110.5$ and $n = 11$, thus $r = 1 - [6(110.5)/11(11^2 - 1)] = .498$.

# 29

# Statistical Decision-Making Using Payoffs

In this section we extend the discussion of statistical decision-making begun in Section 16 by using criteria based on payoffs rather than on error probabilities. For a review of the basic decision model and terminology, the reader should return to Section 15 and Section 16.

### A. Decision Rules, Payoffs, and Act Probabilities.

Example 1 A firm must increase its production capacity to meet an anticipated increase in sales resulting from expansion of the sales force. Two strategies, or acts, for expanding capacity are under consideration. Let these two acts be denoted $C_1$ and $C_2$. $C_1$ would be attractive if the increase in mean weekly sales is small, while $C_2$ would be attractive if the increase is large. The accounting department has prepared the following chart (Figure 29.1) relating to the effect on annual profits of $C_1$ and $C_2$, for various levels of mean weekly sales ($\mu$). The relevant range of $\mu$ values is from 1.0 to 1.5 (thousand units).

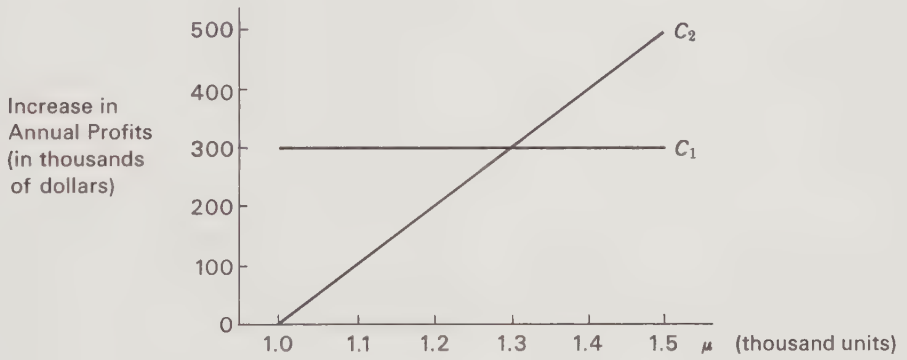



**Figure 29.1**

The firm will use a decision procedure based on the mean weekly sales ($\overline{X}$) for the first four weeks of operation under the new sales program for choosing between $C_1$ and $C_2$. It is known from past experience that the weekly sales represent a random sample from a normal population with standard deviation $\sigma = .20$, and that future weekly sales will be at the same level as sales immediately after the new sales program has begun.

The decision rule which the firm is considering is: If $\overline{X} \leqslant 1.2$, conclude $C_1$; if $\overline{X} > 1.2$, conclude $C_2$. Let this decision rule be denoted $A$.

The act probabilities for decision rule $A$ for various possible values of $\mu$ are shown in Table 29.1. (For a review of the calculation of act probabilities see Section 16, Topic D. Note that since the population is normal with $\sigma = .20$, the sampling distribution of $\overline{X}$ is normal regardless of sample size, with standard deviation $\sigma_{\overline{X}} = .20/\sqrt{4} = .10$.)

Table 29.1

| | Decision Rule $A$ | | Decision Rule $B$ | |
|---|---|---|---|---|
| $\mu$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ |
| 1.0 | .9772 | .0228 | .99865 | .00135 |
| 1.1 | .8413 | .1587 | .9772 | .0228 |
| 1.2 | .5000 | .5000 | _____ | _____ |
| 1.3 | .1587 | .8413 | .5000 | .5000 |
| 1.4 | .0228 | .9772 | .1587 | .8413 |
| 1.5 | .00135 | .99865 | _____ | _____ |

**Problem 1**

Another decision rule which the firm is considering is: If $\overline{X} \leqslant 1.3$, conclude $C_1$; if $\overline{X} > 1.3$, conclude $C_2$. Let this decision rule be denoted $B$. The act probabilities for decision rule $B$ for various possible values of $\mu$ are shown in Table 29.1. Complete the table. √

**Problem 2**

Refer to Example 1.

(a) If $\mu$ actually were 1.1, which act has the higher payoff? _____

(b) If $\mu$ were 1.1, which decision rule ($A$ or $B$) would be the better one? _____

(c) Is the same decision rule as in (b) better if $\mu$ were 1.4? _____√

**B. Expected Payoff Function.** The expected payoff function for a statistical decision rule shows, for every possible outcome state, the expected value of the payoff with that rule.

Example 2 Refer to Example 1. The graph in Example 1 shows, for each value of $\mu$, the payoffs for act $C_1$ and for act $C_2$. These payoffs are shown in the table below for various possible values of $\mu$. Which of the two payoffs for any value of $\mu$ is actually achieved depends on the decision rule utilized and the sample mean $\overline{X}$ obtained. The act probabilities that decision rule $A$ will lead to $C_1$ and $C_2$ respectively for different values of $\mu$ were calculated in Example 1 and are also shown in Table 29.2.

**Table 29.2**

| Outcome State $\mu$ | Payoff | | Rule $A$ Act Probability | | Expected Payoff |
|---|---|---|---|---|---|
| | Act $C_1$ | Act $C_2$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | |
| 1.0 | 300 | 0 | .9772 | .0228 | 293.16 |
| 1.1 | 300 | 100 | .8413 | .1587 | 268.26 |
| 1.2 | 300 | 200 | .5000 | .5000 | 250.00 |
| 1.3 | 300 | 300 | .1587 | .8413 | 300.00 |
| 1.4 | 300 | 400 | .0228 | .9772 | _____ |
| 1.5 | 300 | 500 | .00135 | .99865 | _____ |

Thus, for any value of $\mu$ the payoff is a random variable with a probability distribution. For instance if $\mu = 1.0$, the probability distribution of the payoff is:

| Act Selected | Payoff | Probability |
|---|---|---|
| $C_1$ | 300 | .9772 |
| $C_2$ | 0 | .0228 |

The expected payoff from using decision rule $A$ when $\mu = 1.0$ is the expected value of the above probability distribution, namely:

$$\text{(expected payoff when } \mu = 1.0) = 300(.9772) + 0(.0228) = 293.16.$$

The expected payoffs for other values of $\mu$ are calculated in a similar fashion and are shown in Table 29.2.

**Problem 3**

(a) Complete the expected payoffs for decision rule $A$ in Table 29.2.
(b) For what value of $\mu$ is there indifference whether act $C_1$ or act $C_2$ is chosen? _____
(c) Compare the expected payoff amount with the payoffs for acts $C_1$ and $C_2$, for each value of $\mu$. What generalization does this suggest? _____

_____

_____ √

**Problem 4**

Calculate the expected payoffs for decision rule $B$, using the results of Problem 1, for the first three lines in Table 29.3.

**Table 29.3**

| Outcome State $\mu$ | Payoff | | Rule $B$ Act Probability | | Expected Payoff |
|---|---|---|---|---|---|
| | Act $C_1$ | Act $C_2$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | |
| 1.0 | | | | | |
| 1.1 | | | | | |
| 1.2 | | | | | √ |
| 1.3 | | | | | |
| 1.4 | | | | | |
| 1.5 | | | | | |

**Problem 5**

Complete the last three lines of Table 29.3. √

Example 3 Plotting the expected payoffs for decision rule $A$ on the $Y$ axis corresponding to the possible values of $\mu$ located on the $X$ axis yields the graph of the *expected payoff function* for decision rule $A$:

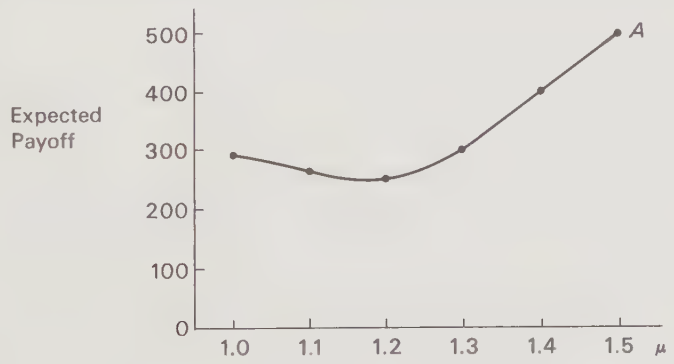



**Figure 29.2**

Answers

1. $\sigma_{\bar{X}} = .20/\sqrt{4} = .10$.

| $\mu$ | $z$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ |
|---|---|---|---|
| 1.2 | +1 | .8413 | .1587 |
| 1.5 | −2 | .0228 | .9772 |

2. (a) When $\mu = 1.1$, act $C_1$ leads to the larger payoff (300 versus 100 for act $C_2$).
   (b) Since decision rule $B$ has the larger probability of concluding $C_1$ when $\mu = 1.1$ (.9772 versus .8413 for decision rule $A$), it follows that decision rule $B$ is the better one when $\mu = 1.1$.
   (c) No, decision rule $A$ is better when $\mu = 1.4$.

3. (a)

| $\mu$ | Expected Payoff |
|---|---|
| 1.4 | 300(.0228) + 400(.9772) = 397.72 |
| 1.5 | 300(.00135) + 500(.99865) = 499.73 |

   (b) $\mu = 1.3$.
   (c) For each value of $\mu$, the expected payoff is an amount between the payoff for act $C_1$ and the payoff for act $C_2$.

4.

| Outcome State | Payoff | | Act Probabilities | | Expected |
|---|---|---|---|---|---|
| $\mu$ | Act $C_1$ | Act $C_2$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | Payoff |
| 1.0 | 300 | 0 | .99865 | .00135 | 299.60 |
| 1.1 | 300 | 100 | .9772 | .0228 | 295.44 |
| 1.2 | 300 | 200 | .8413 | .1587 | 284.13 |

**If you had no trouble with Problem 4, fill in the last three lines of Table 29.3 from the answer to Problem 5. If you had trouble, work Problem 5.**

5.

| Outcome State | Payoff | | Act Probabilities | | Expected |
|---|---|---|---|---|---|
| $\mu$ | Act $C_1$ | Act $C_2$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | Payoff |
| 1.3 | 300 | 300 | .5000 | .5000 | 300.00 |
| 1.4 | 300 | 400 | .1587 | .8413 | 384.13 |
| 1.5 | 300 | 500 | .0228 | .9772 | 495.44 |

**Problem 6**

Sketch the expected payoff function for decision rule $B$ on Figure 29.2, which contains the expected payoff function for decision rule $A$. √

Example 4 One decision rule is said to *dominate* another if for each possible outcome state the first rule yields an expected payoff at least as good as that of the second rule and yields a higher expected payoff under at least one outcome state. A decision rule is *admissible* if it is not dominated.

**Problem 7**

Refer to Figure 29.2.
(a) If $\mu = 1.1$, which decision rule has the higher expected payoff? _____
(b) If $\mu = 1.4$, which decision rule has the higher expected payoff? _____
(c) Are both decision rules admissible? _____ √

(d) What would be the comparative positions of the two curves if rule $A$ dominated rule $B$? _____

_____

_____ √

**C. Criteria for Choosing "Best" Decision Rule.** The decision maker wishes to choose the "best" decision rule from among the many available. (Here, for simplicity, we have only considered two decision rules, $A$ and $B$.) Generally, any decision rule dominated by another is rejected at the outset, but there may still be many admissible decision rules to be considered. When the decision maker is uncertain about which value of $\mu$ will prevail, the choice of the "best" statistical decision rule corresponds to decision-making under conditions of uncertainty, discussed in Section 15, Topic C. The decision criteria considered there are relevant here for selecting the "best" of the admissible decision rules. These criteria are illustrated below, for the simplified case where only two competing rules are considered.

Example 5

*Maximax criterion.* According to this criterion, that decision rule is best which has the largest expected payoff possible in the range of relevant outcome states.

Consider the expected payoff functions for decision rules $A$ and $B$ displayed in Figure 29.2. The relevant values of $\mu$ are from 1.0 to 1.5. In this range, rule $A$ has a higher maximum expected payoff than rule $B$, namely 499.73 at $\mu = 1.5$, hence it is considered best according to the maximax criterion.

*Maximin criterion.* This criterion considers the smallest expected payoff for every admissible decision rule. That decision rule which has the largest minimum expected payoff is considered best according to the maximin criterion.

By inspection, we see that for rule $A$ the smallest expected payoff occurs at $\mu = 1.19$ and equals 249. For rule $B$ it occurs at $\mu = 1.22$ and equals 283. Hence, rule $B$ is considered best according to the maximin criterion because it maximizes the minimum expected payoff.

*Minimax regret criterion.* The regret or opportunity loss for a given decision rule is the difference between the best expected payoff that could be obtained under a given outcome state with the rules under consideration and the expected payoff of the particular decision rule under this outcome state. The minimax regret criterion considers, for each decision rule, the maximum regret possible under the outcome states. The decision rule which has the smallest maximum regret is considered best according to the minimax regret criterion.

**Problem 8**

The expected payoffs and some of the regrets are shown in Table 29.4 for rules $A$ and $B$:

**Table 29.4**

| | Expected Payoff | | Regret | |
|---|---|---|---|---|
| $\mu$ | $A$ | $B$ | $A$ | $B$ |
| 1.0 | 293.16 | 299.60 | 6.44 | 0 |
| 1.1 | 268.26 | 295.44 | _____ | 0 |
| 1.2 | 250.00 | 284.13 | 34.13 | 0 |
| 1.3 | 300.00 | 300.00 | 0 | 0 |
| 1.4 | 397.72 | 384.13 | _____ | — |
| 1.5 | 499.73 | 495.44 | _____ | — |

(a) Complete the regrets in Table 29.4. √

(b) Considering the outcome states in Table 29.4:

(i) What is the largest regret for rule $A$? _____

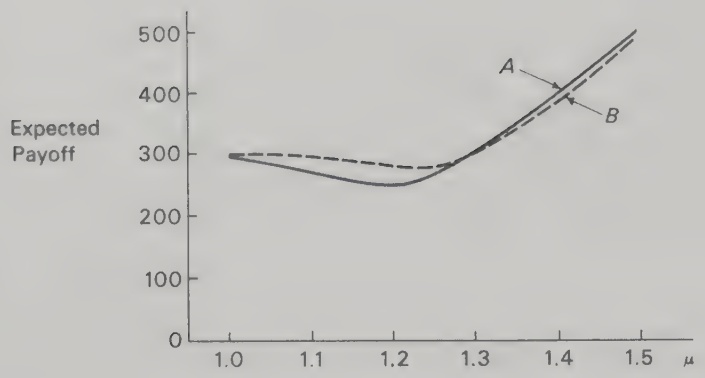
(ii) What is the largest regret for rule $B$? _____

(iii) Which rule is best according to the minimax regret criterion? _____ √

(c) In Figure 29.2, draw a vertical line between the two curves for $\mu = 1.1$. Do the same for $\mu = 1.4$. The vertical line at $\mu = 1.1$ corresponds to the regret for which decision rule? _____ The vertical line at $\mu = 1.4$ corresponds to the regret for which decision rule? _____ √

Answers

6.




7. (a) *B*.

(b) *A*.

(c) Both decision rules are admissible. Note that decision rule *B* leads to larger expected payoffs for values of $\mu$ less than 1.3 and decision rule *A* leads to larger expected payoffs for values of $\mu$ greater than 1.3.

(d) The expected payoff function for rule *A* would never be lower than that for rule *B* and would be higher for at least one point.

8. (a)

| | Expected Payoff | | Regret | |
|---|---|---|---|---|
| $\mu$ | *A* | *B* | *A* | *B* |
| 1.1 | 268.26 | 295.44 | 27.18 | 0 |
| 1.4 | 397.72 | 384.13 | 0 | 13.59 |
| 1.5 | 499.73 | 495.44 | 0 | 4.29 |

(b) (i) **34.13.**
(ii) **13.59.**
(iii) Rule *B* is best because it has the smallest maximum regret.

(c) The vertical line at $\mu = 1.1$ corresponds to the regret for rule *A*. The vertical line at $\mu = 1.4$ corresponds to the regret for rule *B*.

**Problem 9**

Assume that three statistical decision rules, $C$, $D$, and $E$, are being considered for a certain decision problem. Let the expected payoff functions for $C$, $D$, and $E$ be those shown in Figure 29.3. Values of $\mu$ between 1 and 10 are relevant.

(a) Are all three decision rules admissible? _____ √

(b) Which decision rule is considered best according to the following decision criteria? Maximax _____ Maximin _____ Minimax regret _____

(c) If it were certain that $\mu$ is less than 5, which decision rule should be used? _____ √

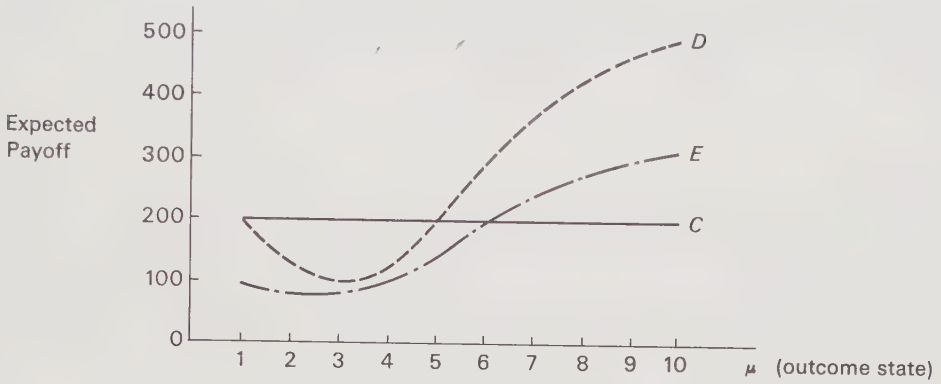



**Figure 29.3**

**Answers**

9. (a) No. Decision rule $E$ is dominated by decision rule $D$ because, for every $\mu$ value in the relevant range, decision rule $D$ leads to a larger expected payoff than decision rule $E$. Hence, $E$ should be disregarded for the remainder of the problem.
   (b) Maximax decision rule is $D$ (largest possible expected payoff is 500 at $\mu = 10$).
   Maximin decision rule is $C$ (largest minimum expected payoff is 200 at every value of $\mu$).
   Minimax regret decision rule is $D$. The maximum regret for decision rule $D$ occurs at $\mu = 3$ and equals $200 - 100 = 100$; the maximum regret for decision rule $C$ occurs at $\mu = 10$ and equals $500 - 200 = 300$.
   (c) If it is certain that $\mu < 5$, decision rule $C$ leads to the largest expected payoff for all values of $\mu$. Hence it is the best rule according to all criteria considered earlier.

## REVIEW PROBLEMS

1. A company has a large number of customers for a new piece of equipment just developed. The company will lease production facilities for a period of time at either of two locations until it can build its own plant. The first location would involve lower manufacturing costs than the second but higher distribution costs. The alternative acts are: $C_1$: lease at location one; $C_2$: lease at location two. The payoff to the company during the lease depends on the mean number of units ($\mu$) sold to each customer during the period. The company has estimated the payoffs for the two acts $C_1$ and $C_2$, as shown in Figure 29.4 for the relevant range of $\mu$ values.

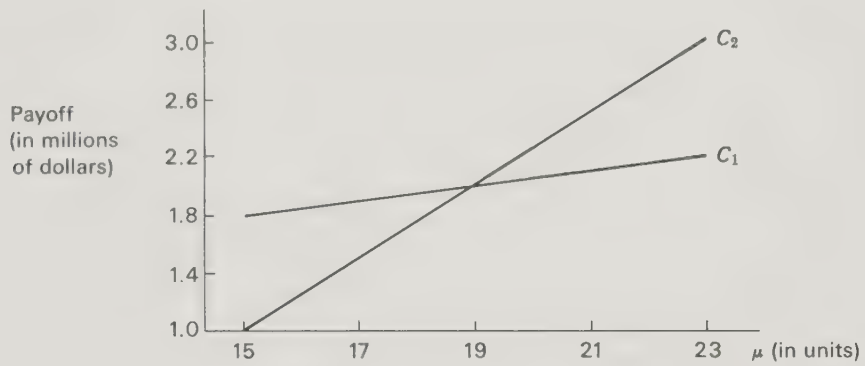



**Figure 29.4**

The company plans to make a decision on the basis of the mean advance order size $\bar{X}$ obtained from a random sample of 64 customers prior to the leasing decision. The following decision rule (rule $A$) is being considered: If $\bar{X} \leqslant 19$, choose $C_1$; if $\bar{X} > 19$, choose $C_2$. It is expected from past experience that the standard deviation of the distribution of number of units sold to each customer is $\sigma = 16$.

(a) For rule $A$, fill in the blanks in Table 29.5.

**Table 29.5**

| | Payoffs | | Rule $A$ Act Probabilities | | |
|---|---|---|---|---|---|
| $\mu$ | $C_1$ | $C_2$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | Expected Payoff |
| 15 | 1.8 | 1.0 | .9772 | .0228 | 1.782 |
| 17 | 1.9 | 1.5 | .8413 | .1587 | 1.837 |
| 19 | 2.0 | 2.0 | .5000 | .5000 | 2.000 |
| 21 | ___ | ___ | ___ | ___ | ___ |
| 23 | ___ | ___ | ___ | ___ | ___√ |

(b) Determine the expected payoffs for the following decision rule (rule $B$):

Whatever the value of $\overline{X}$, choose $C_1$.

| $\mu$: | 15 | 17 | 19 | 21 | 23 |
|---|---|---|---|---|---|
| Expected payoff: | ___ | ___ | ___ | ___ | ___√ |

(c) Plot the expected payoff functions for rules $A$ and $B$ on one graph below (use the results from (a) and (b)). Which decision rule ($A$ or $B$) is best according to each of the following decision criteria? Maximax _____ maximin ____ minimax regret _____ √

## ANSWERS TO REVIEW PROBLEMS

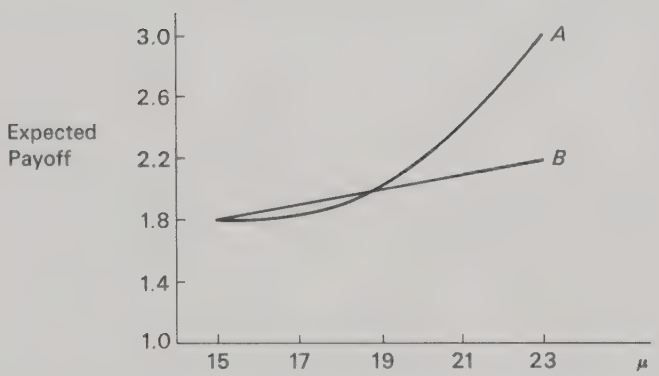
1. (a) $\sigma_{\overline{X}} = 16/\sqrt{64} = 2$. The sampling distribution of $\overline{X}$ is approximately normal in accordance with the Central Limit Theorem.

| | Payoffs | | Act Probabilities | | |
|---|---|---|---|---|---|
| $\mu$ | $C_1$ | $C_2$ | $P(C_1 : \mu)$ | $P(C_2 : \mu)$ | Expected Payoff |
| 21 | 2.1 | 2.5 | .1587 | .8413 | 2.437 |
| 23 | 2.2 | 3.0 | .0228 | .9772 | 2.982 |

(b) Since $C_1$ will be chosen under rule $B$ regardless of the sample information, the expected payoff function for this rule corresponds to the payoff values for act $C_1$ for the different values of $\mu$.

| $\mu$: | 15 | 17 | 19 | 21 | 23 |
|---|---|---|---|---|---|
| Expected payoff: | 1.8 | 1.9 | 2.0 | 2.1 | 2.2 |

(c)




Maximax decision rule is $A$ (largest possible value is 2.982 at $\mu = 23$).
Maximin decision rule is $B$ (largest minimum value is 1.8 at $\mu = 15$).
Minimax regret decision rule is $A$. The maximum regret for decision rule $A$ occurs at $\mu = 17.5$ and equals .068; the maximum regret for decision rule $B$ occurs at $\mu = 23$ and equals .782.

# 30

# Bayesian Decision-Making

In this section we discuss statistical decision-making when probabilities can be assigned to the outcome states prior to obtaining sample information.

**A. Decision-Making When Only Prior Probabilities Are Available.** In Section 29, uncertainty about the true value of the outcome state was assumed; hence criteria for decision-making under uncertainty were employed for determining the best decision rule. At times, however, information about the outcome states is available in the form of probabilities, either from past experience or based on the decision-maker's judgment. These probabilities are called *prior probabilities* and they form a probability distribution for the outcome states which is called a *prior probability distribution*. When the prior probabilities are subjective, it may well occur that two different persons might use different prior probability distributions for the same problem.

When decisions must be made with no sample information, solely on the basis of the prior probabilities, we have the case of decision-making under risk, discussed in Section 15, Topic D. We review this briefly.

Example 1 A mining company is considering the purchase of the mineral rights to a tract of land. An airborne geophysical reconnaissance of the tract indicates three geological anomalies, each of which may or may not be the site of a major body of valuable minerals. The firm must choose between the two acts: $C_1$: do not purchase the mineral rights; $C_2$: purchase the mineral rights.

In this problem, the outcome state affecting the payoffs is the number of major ore bodies among the three sites. Let the outcome state be denoted by $\theta$, and assume that the payoffs for the two acts for the various values of $\theta$ are as given in Table 30.1.

**Table 30.1**

| Number of Major Ore Bodies ($\theta$) | Prior Probability $P(\theta)$ | Payoff of Act ($ thousands) $C_1$ | $C_2$ |
|---|---|---|---|
| 0 | .85 | 0 | −100 |
| 1 | .10 | 0 | 200 |
| 2 | .04 | 0 | 500 |
| 3 | .01 | 0 | 800 |
| | 1.00 | | |

On the basis of extensive experience, company geologists have assigned the prior probabilities shown in Table 30.1 to the four values of $\theta$.

The use of the expected payoff criterion to select the best act was discussed in Section 15. For our example, the expected payoffs for acts $C_1$ and $C_2$ are:

| Act | Expected Payoff |
|---|---|
| $C_1$: | 0(.85) + 0(.10) + 0(.04) + 0(.01) = 0 |
| $C_2$: | −100(.85) + 200(.10) + 500(.04) + 800(.01) = −37 |

Here act $C_1$ yields the largest expected payoff and thus is the best act. The procedure that calls for choosing the act for which the expected payoff is a maximum is often called a *Bayes procedure*, and this optimum act is called the *Bayes act*.

**Problem 1**

(a) For the problem in Example 1, what would be the Bayes act if the prior probabilities were $P(\theta = 0) = .78$, $P(\theta = 1) = .13$, $P(\theta = 2) = .06$, and $P(\theta = 3) = .03$. _____

(b) Comparing the answers in Example 1 and Problem 1 (a) above, can the Bayes act differ for different prior probabilities?_____√

**B. Decision-Making When Prior Probabilities and Sample Information Are Available.** For convenience, we call the case when only prior probabilities and no sample data are available the no-sample case ($n = 0$). When both prior probabilities and sample data are available for decision-making, two steps can be used to find the Bayes act: (1) combine the prior probabilities and the sample data in a suitable fashion to obtain revised prior probabilities (that is, revised in the light of the sample data); and (2) use these revised prior probabilities as in the no-sample case to find the Bayes act. The revision of the prior probabilities is accomplished by means of Bayes' Theorem discussed in Section 5, Topic I. The revised prior probabilities are called the *posterior probabilities* to indicate that they are based both on the prior probabilities and the sample data.

Example 2 Refer to Example 1. Assume the mining company receives permission to select $n = 1$ of the three sites at random and to explore it more closely prior to making a decision on purchase of the tract. Let $X = 1$ denote the sample result that a major ore body is discovered and $X = 0$ that no major ore body is discovered.

Suppose one anomaly is selected at random and the actual result is $X = 0$. The first step in finding the Bayes act is to calculate the posterior (revised) probabilities for the various values of $\theta$, given that $X = 0$. We denote these posterior probabilities $P(\theta \mid X = 0)$.

From Bayes' Theorem, we know that:

$$P(\theta \mid X = 0) = \frac{P(X = 0 \text{ and } \theta)}{P(X = 0)} = \frac{P(X = 0 \mid \theta)\, P(\theta)}{P(X = 0)}$$

and

$$P(X = 0) = P(X = 0 \mid \theta = 0)P(\theta = 0) + P(X = 0 \mid \theta = 1)P(\theta = 1) + P(X = 0 \mid \theta = 2)P(\theta = 2) + P(X = 0 \mid \theta = 3)P(\theta = 3).$$

Here $P(\theta)$ are the prior probabilities given in Example 1. The probabilities $P(X = 0 \mid \theta)$ may be deduced directly from our knowledge of the sampling procedure. For instance, $P(X = 0 \mid \theta = 1) = 2/3$, since if $\theta = 1$, two of the three anomalies do not represent major ore bodies and the probability that the anomaly selected at random will be one of these two is 2/3.

**Problem 2**

Complete the following list of $P(X = 0 \mid \theta)$ probabilities:

$P(X = 0 \mid \theta = 0) =$_____; $P(X = 0 \mid \theta = 1) = 2/3$;
$P(X = 0 \mid \theta = 2) =$_____; $P(X = 0 \mid \theta = 3) =$_____.√

**Table 30.2**

| $\theta$ | $P(\theta)$ | $P(X = 0 \mid \theta)$ | $P(X = 0 \mid \theta)P(\theta)$ | $\frac{P(X = 0 \mid \theta)P(\theta)}{P(X = 0)} = P(\theta \mid X = 0)$ |
|---|---|---|---|---|
| 0 | .85 | 1 | (1) (.85) = .8500 | .8500/.9300 = .914 |
| 1 | .10 | 2/3 | .0667 | .072 |
| 2 | .04 | 1/3 | .0133 | .014 |
| 3 | .01 | 0 | .0000 | .000 |
| Total | 1.00 | | $P(X = 0) = .9300$ | 1.000 |

Example 3 Refer to Example 2. Given the probabilities $P(\theta)$ (Example 1 ) and $P(X = 0 \mid \theta)$ (Problem 2), the formula for Bayes' Theorem may be used to calculate the posterior probabilites, $P(\theta \mid X = 0)$, as in Table 30.2.

**Problem 3**

Refer to Example 3. Suppose that an exploration of the sample anomaly indicated a major ore body, that is, $n = 1, X = 1$.

(a) Obtain the posterior probability distribution for $\theta$ given this sample information:

**Table 30.3**

| $\theta$ | $P(\theta)$ | $P(X = 1 \mid \theta)$ | $P(X = 1 \mid \theta)P(\theta)$ | $P(\theta \mid X = 1)$ | |
|---|---|---|---|---|---|
| 0 | .85 | 0 | _____ | _____ | |
| 1 | .10 | 1/3 | _____ | _____ | |
| 2 | ___ | ___ | _____ | _____ | |
| 3 | ___ | ___ | _____ | _____ | |
| Total | ___ | | _____ | _____ | √ |

(b) Compare this posterior distribution with the one in Example 3. Is the posterior distribution affected by the sample result?_____√

Example 4 Refer to Example 3. Once the posterior probability distribution for $\theta$, given the sample information $X = 0$, has been obtained, the expected payoff for each act is calculated, using the posterior probabilities. The act that has the maximum expected payoff is the Bayes act. The expected payoff calculations are (act payoffs are given in Example 1):

| Act | | Expected Payoff |
|---|---|---|
| $C_1$: | $0(.914) + 0(.072) + 0(.014) + 0(.000) =$ | 0 |
| $C_2$: | $-100(.914) + 200(.072) + 500(.014) + 800(.000) =$ | −70 |

Act $C_1$ yields the largest expected payoff, and thus is the Bayes act when $n = 1, X = 0$.

**Problem 4**

Refer to Problem 3. Given the sample result $n = 1, X = 1$, calculate the expected payoffs for $C_1$ and $C_2$.
$C_1$:_____ ; $C_2$:_____ . Which is the Bayes act?_____√

**C. Bayes Decision Rule and Expected Value of Bayes Decision Rule.** A *Bayes decision rule* or *Bayes strategy*, for a given sample size, indicates the Bayes act for each possible sample outcome.

**Problem 5**

(a) In the mining company decision problem, for a sample of one anomaly ($n = 1$), it was seen that act $C_1$ is the Bayes act if $X = 0$ and act $C_2$ is the Bayes act if $X = 1$. State the Bayes decision rule for $n = 1$. _____

_____

(b) Since the Bayes act for a given outcome is that act which maximizes the expected payoff, how does the expected payoff of the Bayes decision rule compare with the expected payoff of any alternative decision rule? _____

(c) Might the Bayes decision rule in (a) be changed if the prior probabilities or payoffs were changed?_____√

Answers

1. (a) Act — Expected Payoff

$C_1$: $0(.78) + 0(.13) + 0(.06) + 0(.03) = 0$
$C_2$: $-100(.78) + 200(.13) + 500(.06) + 800(.03) = 2$

Act $C_2$ is the Bayes act.

(b) **Yes, different prior probabilities have led to different answers in Example 1 and (a) above.**

2. $P(X = 0 \mid \theta = 0) = 1$, since if none of the anomalies represent major ore bodies, it is certain that the anomaly sampled will not yield a major ore body. Similarly $P(X = 0 \mid \theta = 2) = 1/3$ and $P(X = 0 \mid \theta = 3) = 0$.

3. (a)

| $\theta$ | $P(\theta)$ | $P(X = 1 \mid \theta)$ | $P(X = 1 \mid \theta)P(\theta)$ | $\frac{P(X = 1 \mid \theta)P(\theta)}{P(X = 1)} = P(\theta \mid X = 1)$ |
|---|---|---|---|---|
| 0 | .85 | 0 | $0(.85) = .0000$ | $(.0000/.0700) = .000$ |
| 1 | .10 | 1/3 | .0333 | .476 |
| 2 | .04 | 2/3 | .0267 | .381 |
| 3 | .01 | 1 | .0100 | .143 |
| | 1.00 | | $P(X = 1) = .0700$ | 1.000 |

(b) **Yes, the different sample results have led to different posterior distributions in Example 3 and (a) above.**

4. Act — Expected Payoff, $X = 1$

$C_1$: $0(.000) + 0(.476) + 0(.381) + 0(.143) = 0$
$C_2$: $-100(.000) + 200(.476) + 500(.381) + 800(.143) = 400.1$

Act $C_2$ is the Bayes act when $n = 1$, $X = 1$.

5. (a) If $X = 0$, select act $C_1$; if $X = 1$, select act $C_2$.

(b) **Since the Bayes act for a given outcome maximizes the expected payoff for that outcome, the Bayes decision rule, which is a combination of Bayes acts, maximizes the expected payoff over any alternative decision rule.**

(c) **The Bayes decision rule may be changed by either changing the prior probabilities or the payoffs or both. The reason is that these changes may change the Bayes acts which in turn determine the Bayes decision rule.**

**Problem 6**

(a) For the mining company decision problem in Example 1, state the Bayes decision rule for the no-sample case. _____

(b) Suppose that in another tract, there are six anomalies and a random sample of three is selected for evaluation. If the Bayes acts for the different possible sample outcomes are as follows, state the Bayes decision rule for $n = 3$:

| $X$ | Bayes Act | _____ |
|---|---|---|
| 0 | $C_1$ | _____ √ |
| 1 | $C_1$ | |
| 2 | $C_2$ | |
| 3 | $C_2$ | |

Example 5 Consider the Bayes decision rule given in Problem 5(a). Table 30.4 summarizes the various results obtained.

**Table 30.4**

| Sample Outcome ($X$) | $P(X)$ | Bayes Act | Expected Payoff of Bayes Act |
|---|---|---|---|
| $X = 0$ | .93 | $C_1$ | 0 |
| $X = 1$ | .07 | $C_2$ | 400.1 |
| | 1.00 | | |

Recall that $P(X = 0) = .93$ and $P(X = 1) = .07$ were calculated in Example 3 and Problem 3 respectively. The expected payoffs for the two Bayes acts were calculated in Example 4 and Problem 4.

The expected payoff from using the Bayes decision rule for $n = 1$ is obtained from the above table as follows:

$$\text{expected payoff of Bayes decision rule } (n = 1) = 0(.93) + 400.1(.07) = 28.0.$$

Note that this expected payoff must be the maximum for any decision rule based on $n = 1$ since the expected payoff of the Bayes act for $X = 0$ is a maximum for that sample outcome, and similarly the expected payoff of the Bayes act for $X = 1$ is a maximum.

**Problem 7**

Refer to the Bayes decision rule for the no-sample case derived in Problem 6(a). What is the expected payoff of this decision rule? (*Hint*: Use the results from Example 1.) ____ √

**Problem 8**

In Example 5 and Problem 7 respectively, the expected payoffs of the Bayes decision rules for $n = 0$ and $n = 1$ were calculated. In these calculations the cost of sampling was not considered.

(a) Assuming that the cost of investigating each site is \$20 thousand, what will be the expected payoffs of the Bayes decision rules for $n = 0$ and $n = 1$, net of the cost of sampling, that is, after deducting the cost of sampling? _____ _____

(b) If the company must choose between making a decision based on no sample information ($n = 0$) and taking a sample of size one ($n = 1$) as a basis for making a decision, which sample size should the company use? ____ √

**D. Determination of Optimal Sample Size.** The reader may find it helpful to review Section 15, Topic E, before proceeding.

Example 6 We determined above the Bayes decision rules based on samples of size $n = 0$ and $n = 1$, and found the net expected payoffs of these strategies (net of sampling costs) to be 0 and 8.0 respectively. To find the best sample size, one must also consider the net expected payoffs of the Bayes decision rules for other sample sizes.

To save one the trouble of looking at every sample size, the *expected value of perfect information* may be used. Let *EVPI* denote the expected value of perfect information and $c$ the cost of each sample unit. The only sample sizes ($n$) which need to be considered are those for which $cn \leqslant EVPI$. The reason is that for larger sample sizes, the expected payoff net of sampling costs will be less than the expected payoff for $n = 0$.

**Table 30.5**

| $\theta$ | $P(\theta)$ | Optimal Act if $\theta$ Known | Payoff of Optimal Act if $\theta$ Known | Optimal Payoff Times $P(\theta)$ |
|---|---|---|---|---|
| 0 | .85 | $C_1$ | 0 | 0 |
| 1 | .10 | $C_2$ | 200 | 20 |
| 2 | .04 | — | — | — |
| 3 | .01 | — | — | — |
| Total | 1.00 | | | ___ √ |

**Answers**

6. (a) Select act $C_1$.
   (b) If $X \leqslant 1$, select act $C_1$; if $X > 1$, select act $C_2$.

7. The expected payoff is 0, the expected payoff of the Bayes act $C_1$ (see Example 1).

8. (a)

| $n$ | Expected Payoff | Expected Payoff Net of Sampling Cost |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 28.0 | 28.0 − 20 = 8.0 |

   (b) The company should use $n = 1$ because the Bayes decision rule for $n = 1$ leads to a larger expected payoff net of sampling costs (8.0), than that of the Bayes decision rule for $n = 0$.

**Problem 9**

(a) Refer to Example 1. Complete Table 30.5 to calculate the expected payoff with perfect (free) information.
(b) In Example 1, it was found that the expected payoff of the Bayes act for $n = 0$ is 0. Hence, what is the expected value of perfect information? $EVPI =$ _____
(c) Using the information on $EVPI$ in (b), what sample sizes need not be considered in the mining company decision problem, given that the unit cost of sampling is $c = \$20$ thousand? _____√

Example 7 In Problem 9(c) it was found that the firm need not consider sample size $n = 3$. The cases where $n = 0$ and $n = 1$ have been investigated, and for $n = 2$, the Bayes decision rule is: If $X = 0$, select act $C_1$; if $X = 1$ or 2, select act $C_2$. The expected payoff for this decision rule, net of sampling costs, is \$1.3 thousand (calculations not shown).

**Problem 10**

(a) Compare the net expected payoff for $n = 2$ with those for $n = 0$ and $n = 1$ (Problem 8(a)). Which is the optimal sample size? _____
(b) What decision rule should be used for the sample result based on the optimal sample size? _____
_____
(c) What is the expected payoff, net of sampling costs, from using this decision rule? _____√
(d) How could the results for $n = 0$, $n = 1$, and $EVPI$ have been used to exclude the case $n = 2$ from consideration? (*Hint*: For $n = 2$, sampling cost is 40, $EVPI = 48$, and expected net payoff for $n = 1$ is 8.) _____
_____
_____√

## REVIEW PROBLEMS

1. A used car dealer is considering whether to purchase ($C_1$) or not purchase ($C_2$) a lot of four cars being retired from the fleet of a taxicab company. The number of cars in the lot which require a major overhaul is the key determinant of the payoff that the dealer will realize if he makes the purchase. If he rejects the lot, the payoff is zero in any case. The dealer is able to assign prior probabilities on the number of cars in the lot requiring major overhauls on the basis of judgment and experience. Shown in Table 30.6 are the dealer's specific estimates of the prior probabilities and payoffs.

**Table 30.6**

| Number of Cars Requiring Major Overhaul | Prior Probability | Payoff of Act | |
|---|---|---|---|
| | | $C_1$ | $C_2$ |
| 0 | .30 | 900 | 0 |
| 1 | .40 | 500 | 0 |
| 2 | .20 | 100 | 0 |
| 3 | .08 | −300 | 0 |
| 4 | .02 | −700 | 0 |

The dealer has the option of having one or more of the cars selected at random and taken to an automobile service station for an inspection. The cost of inspecting a car is \$20.

(a) If no cars are inspected by the dealer, what act should he select according to the Bayes procedure? _____ What is the expected payoff of the Bayes act? _____

(b) What is the maximum number of cars which the dealer would want to have inspected? (*Hint*: Use *EVPI*; $c = \$20$.) _____ √

(c) If $n = 1$, the Bayes decision rule is: Select act $C_1$ whether $X = 0$ or $X = 1$. The expected payoff, before sampling costs, for this decision rule is \$452. Find the expected payoff, net of sampling costs, for this decision rule. _____

(d) What is the optimal sample size? _____ What is the Bayes decision rule for this sample size? _____ √

Answers

9. (a)

| $\theta$ | $P(\theta)$ | Optimal Act if $\theta$ Known | Payoff of Optimal Act if $\theta$ Known | Optimal Payoff Times $P(\theta)$ |
|---|---|---|---|---|
| 0 | .85 | $C_1$ | 0 | 0 |
| 1 | .10 | $C_2$ | 200 | 20 |
| 2 | .04 | $C_2$ | 500 | 20 |
| 3 | .01 | $C_2$ | 800 | 8 |
| | 1.00 | | | 48 |

(b) $EVPI = 48 - 0 = \$48$ thousand.

(c) With $EVPI = 48$ and $c = 20$, the mining company should consider all sample sizes $n$ for which $20n \leqslant 48$. Hence, the sample sizes $n = 0$, $n = 1$, and $n = 2$ should be investigated, and $n = 3$ need not be considered.

10. (a) $n = 1$.

(b) If $X = 0$, select act $C_1$; if $X = 1$, select act $C_2$.

(c) 8.0.

(d) Because $EVPI = 48$ and the sampling cost is 40, the expected payoff, net of sampling costs, for $n = 2$ cannot exceed 8. Since 8 is the expected net payoff for $n = 1$, it follows that at best you are equally well off with a sample of two as with a sample of one, but you may be worse off. Hence, sample size two need not be considered.

## ANSWERS TO REVIEW PROBLEMS

1. (a) Act — Expected Payoff

$C_1$: $900(.30) + 500(.40) + 100(.20) - 300(.08) - 700(.02) = 452$

$C_2$: $0(.30) + 0(.40) + 0(.20) + 0(.08) + 0(.02) = 0$

Act $C_1$ is the Bayes act for $n = 0$. It has an expected payoff of \$452.

(b) Expected payoff with perfect information $= 900(.30) + 500(.40) + 100(.20) + 0(.08) + 0(.02) = 490$. Expected payoff of the Bayes act with only prior probabilities = 452 (see the answer to (a) above). Thus, $EVPI = 490 - 452 = 38$. The cost of inspecting each car is \$20, thus $c = 20$. Since $n = 1$ is the largest value of $n$ for which $20n \leqslant 38$, it follows that, at most, the dealer would want to have only one car inspected.

(c) The expected payoff, net of sampling costs, is $452 - 20 = \$432$.

(d)

| Sample Size $n$ | Net Expected Payoff from Using the Bayes Decision Rule | Bayes Decision Rule |
|---|---|---|
| 0 | 452 | Select Act $C_1$ |
| 1 | 432 | Select Act $C_1$ |

Since $452 > 432$, the dealer should not have any cars inspected. The Bayes decision rule is to select act $C_1$, purchase the lot.

# 31

# Normal Prior Distribution and Normal Sampling

In this section, we consider Bayesian decision-making concerning a population mean $\mu$ when the prior distribution is normal.

### A. Decision-Making When Only Prior Probabilities Are Available.

Example 1 Consider again Example 1 of Section 29, where a firm has strengthened its sales force, and as a result expects its sales to increase. To meet the anticipated new demand, the firm's production capacity must be increased. Two strategies or acts for expanding capacity are under consideration. One act, $C_1$, would be attractive if the increase in mean weekly sales is small, while the other, $C_2$, would be attractive if the increase is large. The accounting department has prepared payoff functions, showing the effect on annual profits of $C_1$ and $C_2$ for various possible levels of mean weekly sales, $\mu$, as shown in Figure 31.1.

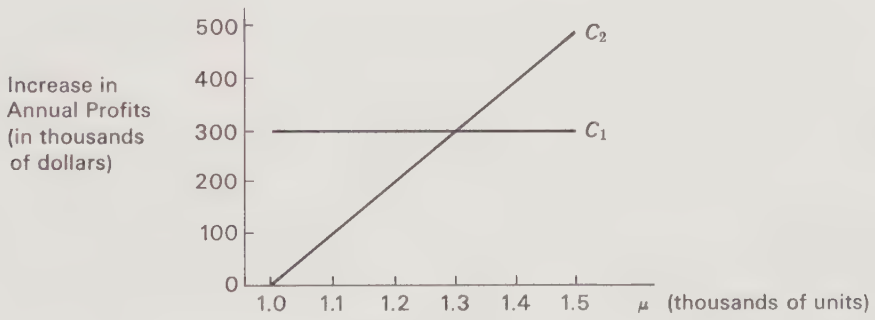



**Figure 31.1**

Specifically, the two payoff curves are:

$$P_1 = 300,$$
$$P_2 = -1000 + 1000\,\mu.$$

The firm is uncertain about the new level of mean weekly sales ($\mu$) which will result, but the sales manager, on the basis of judgment and experience, has assigned a normal prior probability distribution to $\mu$, which has a mean $M_0 = 1.2$ ($ thousands) and standard deviation $S_0 = .30$ ($ thousands).

To find the best act if no sampling is done, we determine the expected payoff for each act. For act $C_1$, we have:

$$E(P_1) = E(300) = 300.$$

For act $C_2$, we have:

$$E(P_2) = E(-1000 + 1000\,\mu) = -1000 + 1000\,E(\mu)$$
$$= -1000 + 1000\,(1.2) = 200.$$

Here $E(\mu) = M_0 = 1.2$ according to the sales manager's prior distribution. Since the expected payoff for $C_1$ is the higher, that is the best act.

Note that for the no-sample case, the fact that the prior distribution is normal did not enter explicitly. This is a consequence of the linear payoff functions, where only $E(\mu)$ is required, and $E(\mu) = M_0$ regardless of the nature of the prior distribution.

**Problem 1**

Refer to Example 1. Suppose the two payoff curves are $P_1 = 400 - 200\,\mu$ and $P_2 = -300 + 500\,\mu$. If the sales manager assigns a normal prior distribution to $\mu$ with mean $M_0 = 1.3$:

(a) Find: $E(P_1) =$ _____ ; $E(P_2) =$ _____

(b) Which is the best act?_____√

**B. Expected Value of Perfect Information.** We consider now the expected value of perfect information (discussed in Section 15, Topic E) when the prior distribution is normal and the payoff functions are linear.

Example 2 To find the expected value of perfect information (*EVPI*) conveniently in this case, it is necessary to convert the linear payoff functions shown above ($P_1 = 300, P_2 = -1000 + 1000\,\mu$) into *linear opportunity loss functions*. The opportunity loss from selecting act $C_1$ at a particular value of $\mu$ is the difference between the maximum payoff which may be realized from the act optimal at that value of $\mu$ and the payoff for act $C_1$ at that value of $\mu$. For instance, at $\mu = 1.5$, the maximum payoff is obtained by selecting act $C_2$ and realizing 500. Act $C_1$ has a payoff of 300 at $\mu = 1.5$. Thus the opportunity loss from selecting act $C_1$ at $\mu = 1.5$ is $500 - 300 = 200$. For any $\mu$ less than 1.3, the maximum payoff and the payoff from act $C_1$ are identically 300. Hence, there is no opportunity loss from selecting act $C_1$ for any $\mu$ less than 1.3.

The opportunity loss function for act $C_1$ is denoted by $OL_1$, and is:

$$OL_1 = 0 \quad \text{for } \mu < 1.3$$
$$= -1300 + 1000\,\mu \quad \text{for } \mu \geqslant 1.3.$$

A graph of $OL_1$ is given in Figure 31.2.

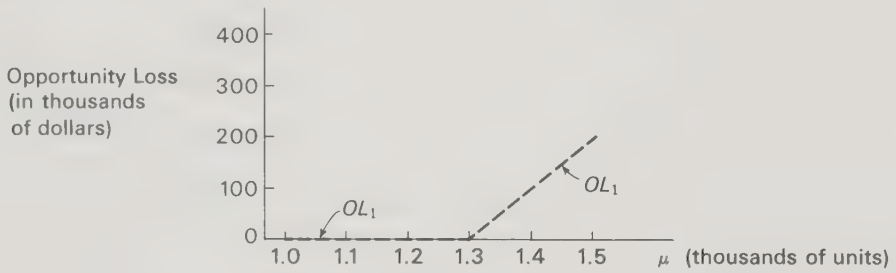



**Figure 31.2**

**Problem 2**

(a) Find the opportunity loss of act $C_2$ when: $\mu = 1.0$ _____; $\mu = 1.2$ _____; $\mu = 1.5$ _____

(b) Obtain $OL_2$ in symbolic form. _____

_____√

(c) On the opportunity loss function graph in Example 2, plot the opportunity loss function for act $C_2$.

(d) The value of $\mu$ for which the opportunity loss is zero for both acts is called the *indifference* or *break-even point* and will be denoted by $K$. Here, $K =$ _____

(e) For what value of $\mu$ are the two payoff functions $P_1$ and $P_2$ equal? (See Example 1.) $\mu =$ _____ Compare this value of $\mu$ with the result in (d). What generalization does this suggest? _____

_____

_____

(f) What is the absolute value of the slope of the opportunity loss function $OL_1$ in the region where it is not horizontal? _____ For $OL_2$? _____ What generalization does this suggest? _____ _____ The absolute value of the slope will be denoted by $t$. √

Example 3 When the prior probability distribution of $\mu$ is normal and the payoff functions for the two acts are linear (as they are in Example 1), the expected value of perfect information (*EVPI*) is given by:

$$EVPI = tSL(D) \quad \text{where } D = |(K - M)/S|.$$

Here $M$ and $S$ are the mean and standard deviation respectively of the prior distribution of $\mu$. The symbol $t$ denotes the absolute value of the slope for the opportunity loss functions of the two acts, and $K$ denotes the indifference point. $D$ is the *absolute* value of the deviation of the indifference point $K$ from the mean $M$, relative to $S$. $L(D)$ is the value of the unit normal loss function at $D$, which is tabulated in Table A-9. For instance, when $D = 1.32$, Table A-9 indicates that $L(D) = L(1.32) = .04363$.

**Problem 3**

In Table A-9, find $L(.05) =$ _____; $L(2.01) =$ _____ √

Example 4 In the decision problem described in Example 1, the firm wishes to calculate the expected value of perfect information. Since *EVPI* provides an upper bound on the amount which should be expended for any information, this is a helpful consideration for the firm in deciding whether to obtain sample information before making a decision and, if so, how much sample information should be obtained.

To calculate the expected value of perfect information for the example considered here, we note that: $M = M_0 = 1.2$ and $S = S_0 = .30$ (see Example 1), $t = 1000$ and $K = 1.3$ (see Problem 2), and hence $D = |(1.3 - 1.2)/.30| = .33$. From Table A-9, we see that $L(D) = L(.33) = .2555$. Substituting the above values into the formula for *EVPI* given above, we obtain $EVPI = 1000\ (.30)\ (.2555) = 76.65$, that is, $EVPI = \$76{,}650$.

**Problem 4**

Suppose the two payoff curves are $P_1 = 400 - 200\,\mu$ and $P_2 = -300 + 500\mu$, and the mean and standard deviation of the normal prior probability distribution of $\mu$ are $M_0 = 1.3$ and $S_0 = .20$ respectively.

(a) Find: $EVPI =$ _____ √

(b) Would perfect information be valuable here? _____

(c) What happens to *EVPI*, the further apart are $K$ and $M$? _____ _____ √

**C. Determining the Posterior Probability Distribution.** As before, we assume that the prior probability distribution for $\mu$ is normal, with mean $M_0$ and standard deviation $S_0$. Suppose now that a random sample of $n$ observations has been obtained and the sample mean $\overline{X}$ calculated. We assume that the sample was selected from a population with mean $\mu$ and known standard deviation $\sigma$. (If $n$ is large, the sample standard deviation $s$ may be used to approximate $\sigma$ when the latter is unknown.) We further assume that the sampling distribution of $\overline{X}$ is normal. (When $n$ is small, this requires that the population sampled is normal. When $n$ is large, the Central Limit Theorem assures approximate normality of the sampling distribution of $\overline{X}$ regardless of the nature of the population sampled.)

Under these conditions, the posterior probability distribution of $\mu$ is normal, with mean $M_1$ and standard deviation $S_1$, where $M_1$ and $S_1$ are given by:

$$M_1 = \frac{\dfrac{M_0}{S_0^2} + \dfrac{\overline{X}}{\sigma^2/n}}{\dfrac{1}{S_0^2} + \dfrac{1}{\sigma^2/n}} \quad \text{and} \quad \frac{1}{S_1^2} = \frac{1}{S_0^2} + \frac{1}{\sigma^2/n}.$$

The posterior distribution thus represents a combination of the prior probabilities and the sample information.

Answers

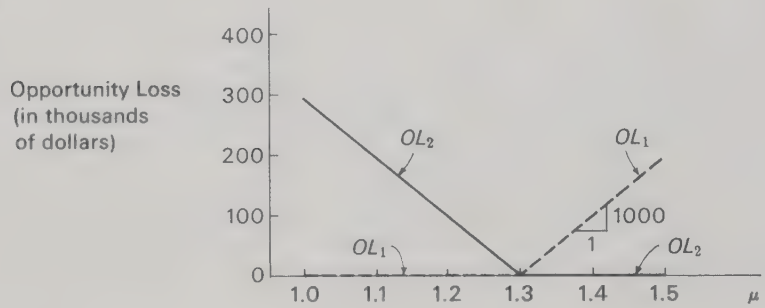
1. (a) $E(P_1) = 400 - 200\,E(\mu) = 400 - 200(1.3) = 140$;
   $E(P_2) = -300 + 500\,E(\mu) = -300 + 500(1.3) = 350$.
   (b) Since $E(P_1) = 140$ and $E(P_2) = 350$, $C_2$ is the best act.

2. (a)

| $\mu$ | Opportunity Loss of Act $C_2$ |
|---|---|
| 1.0 | $300 - 0 = 300$ |
| 1.2 | $300 - 200 = 100$ |
| 1.5 | $500 - 500 = 0$ |

   (b) $OL_2 = 1300 - 1000\mu$ for $\mu < 1.3$
   $= 0$ for $\mu \geqslant 1.3$.

   (c)




   (d) $K = 1.3$.
   (e) $\mu = 1.3$. The value of $\mu$ for which the two payoff functions are equal is equal to the value of $\mu$ for which both opportunity losses are zero, which is the indifference point $K$.
   (f) Both opportunity loss functions have the same absolute slope. In this case, $t = 1000$ ($ thousands per one thousand units).

3. $L(.05) = .3744$; $L(2.01) = .0^2 8266 = .008266$.

4. (a) Here $t = 700$, $S_0 = .20$, $K = 1.0$, $M_0 = 1.3$,
   $D = |(1.0 - 1.3)/.20| = 1.50$, and $L(D) = L(1.50) = .02931$.
   Hence: $EVPI = 700(.20)(.02931) = 4.10$ or $4100.
   (b) Perfect information is relatively less valuable here than it is in Example 4.
   (c) The further apart $K$ and $M$ are, the larger is $D$, the smaller is $L(D)$, and the smaller is $EVPI$. This may be seen from the formula for $EVPI$ and Table A-9.

Example 5 Refer to Example 1, where the sales manager has assigned a normal prior probability distribution to $\mu$ which has mean $M_0 = 1.2$ and standard deviation $S_0 = .30$ ($ thousands). Suppose that the firm has decided to obtain sample information about the value of $\mu$ by utilizing the mean weekly sales ($\overline{X}$) for the first four weeks of operation under the new sales organization. The weekly sales amounts are assumed to represent a random sample from a normal population; thus the sampling distribution of $\overline{X}$ is normal. The population standard deviation is known to be $\sigma = .20$.

Suppose $\overline{X} = 1.3$. What is the posterior distribution for $\mu$? Since $M_0 = 1.2$, $S_0 = .30$, $n = 4$, $\sigma = .20$, and $\overline{X} = 1.3$, the posterior distribution is a normal distribution with mean and standard deviation of:

$$M_1 = \frac{\dfrac{1.2}{(.30)^2} + \dfrac{1.3}{(.2)^2/4}}{\dfrac{1}{(.30)^2} + \dfrac{1}{(.2)^2/4}} = 1.290 \quad \text{and} \quad \frac{1}{S_1^2} = \frac{1}{(.30)^2} + \frac{1}{(.2)^2/4} = 111.11, \text{ or } S_1 = .0949.$$

**Problem 5**

(a) Refer to Example 5. Sketch the prior and posterior probability distributions for $\mu$ on the scale shown below:

(b) Does the posterior distribution suggest that mean weekly sales are likely to be higher than was expected prior to obtaining the sample information? _____ √

**Problem 6**

Refer to Example 5. Suppose mean weekly sales for the first *nine* weeks of operation were $\bar{X} = 1.4$ thousand units. What is the form of the posterior distribution for $\mu$ in this case? $M_1$ = _____ ; $S_1$ = _____ √

Example 6 Once sample information has been obtained, the expected value of perfect information is obtained in the same manner as prior to the sample information, except that the posterior mean $M_1$ and posterior standard deviation $S_1$ are used in the formula for *EVPI*.

**Problem 7**

(a) In Example 5, the posterior distribution for $\mu$, based on mean weekly sales for a sample of four weeks, was developed. Values of $M_1 = 1.290$ and $S_1 = .0949$ were obtained. Calculate the expected value of perfect information given this posterior distribution. (The opportunity loss functions and other relevant information were given in Example 2 and Problem 2.) *EVPI* = _____ √

(b) Is the expected value of perfect information as large now as it was before the sample information was received? (Previously *EVPI* = $76,650; see Example 4.) _____

(c) Is it reasonable that *EVPI* declines after sample information has been obtained? _____ √

**D. Expected Value of Sample Information.** The decision-maker can determine the expected value of sample information (*EVSI*) prior to its collection in order to assess the feasibility of sampling. The calculation is:

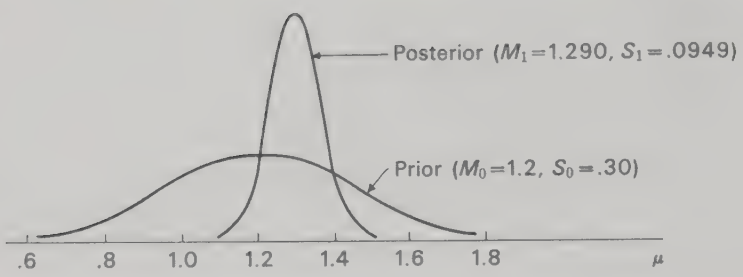
$$EVSI = tS_*L(D_*), \quad \text{where } D_* = |(K - M_0)/S_*| \text{ and } S_* = \sqrt{S_0^2 - S_1^2}.$$

Recall that:

$$\frac{1}{S_1^2} = \frac{1}{S_0^2} + \frac{1}{\sigma^2/n}.$$

**Answers**

5. (a)




(b) Yes, since $M_1$ is greater than $M_0$. Nevertheless, the posterior distribution still exhibits a wide range of possible values of $\mu$.

6.
$$M_1 = \frac{\dfrac{1.2}{(.30)^2} + \dfrac{1.4}{(.20)^2/9}}{\dfrac{1}{(.30)^2} + \dfrac{1}{(.20)^2/9}} = 1.39;$$

$$\frac{1}{S_1^2} = \frac{1}{(.30)^2} + \frac{1}{(.20)^2/9} = 236.1 \text{ or } S_1 = .0651.$$

Thus, the posterior distribution for $\mu$ is a normal distribution with mean 1.39 and standard deviation .0651.

7. (a) Here $t = 1000$, $S_1 = .0949$, $K = 1.3$, $M_1 = 1.290$, $D = |(1.3 - 1.290)/.0949| = .11$, and $L(D) = L(.11) = .3464$. Hence: $EVPI = 1000(.0949)(.3464) = \$32.9$ thousand.

(b) The expected value of perfect information has declined from \$76,650 to \$32,900 with the receipt of the sample information.

(c) Yes, it is intuitively reasonable since now additional information has been obtained. However, in some special circumstances *EVPI* can increase.

**Problem 8**

How does *EVSI* change as the sample size is increased?_____√

Example 7 Refer to Example 5. Let us determine the expected value of sample information based on $n = 4$. Of course, this expected value refers to the value of the information prior to the taking of the sample.

For this example, $M_0 = 1.2, S_0 = .30, t = 1000, K = 1.3, \sigma = .20$, and $n = 4$. Substituting these values into the expressions given above yields:

$$\frac{1}{S_1^2} = \frac{1}{(.30)^2} + \frac{1}{(.20)^2/4} = 111.11 \quad \text{or} \quad S_1 = .0949.$$

(This result was obtained in Example 5.)

Hence, $S_* = \sqrt{(.30)^2 - (.0949)^2} = .285, D_* = |(1.3 - 1.2)/.285| = .35, L(D_*) = L(.35) = .2481$, and $EVSI = 1000\,(.285)\,(.2481) = 70.7$. Thus, the expected value of the sample information is \$70,700.

**Problem 9**

Refer to Example 7.

(a) Find the expected value of sample information based on the weekly sales figure for the first week ($n = 1$) of operation under the new sales program. $EVSI =$ _____ √

(b) Does the expected value of sample information in (a) represent a substantial portion of the expected value of perfect information for this decision problem? ($EVPI = \$76{,}650$; see Example 4.) _____ _____ What are the implications of this with regard to the sample size to be used? _____ _____ √

**E. Sampling Costs and Optimal Sample Size.** A comparison of the cost of sampling with the expected value of sample information for different sample sizes indicates which sample size will lead to the greatest expected net gain from sampling. If we define *ENGS* as the expected net gain from sampling, then $ENGS = EVSI - cn$. Here $c$ is the cost of obtaining each sample observation.

Example 8 Refer to Example 5. Suppose the firm needs to decide in advance how many weeks it will wait prior to making the decision. Assume that every week the firm defers making a decision with respect to the acts $C_1$ and $C_2$, it loses \$10,000. Thus $c = \$10$ thousand in this problem.

Recall that in Example 7 it was shown that $EVSI = \$70.7$ thousand when $n = 4$ and in Problem 9(a) it was shown that $EVSI = \$57.6$ thousand when $n = 1$. By definition, $EVSI = 0$ when $n = 0$. Also the expected value of perfect information is an upper bound on *EVSI*, so that when $n$ is large, *EVSI* is approximately equal to *EVPI*, \$76.65 thousand in this case.

A graph of *EVSI* and sampling costs is shown in Figure 31.3.

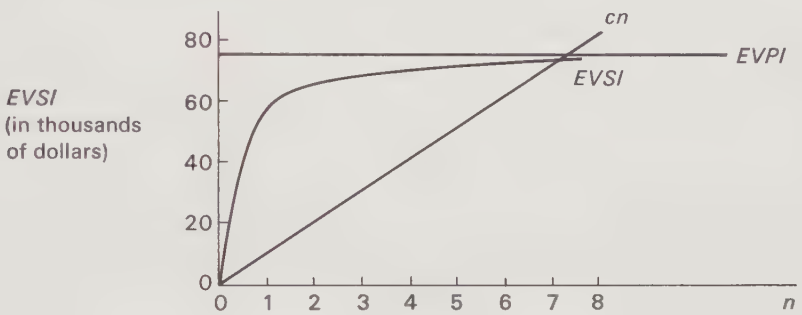



**Figure 31.3**

**Problem 10**

Refer to Example 8.

(a) Determine *ENGS* for the following values of $n$:

| $n$ | *EVSI* | $cn$ | *ENGS* |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 57.6 | 10 | — |
| 2 | 65.4 | 20 | — |
| 3 | 68.5 | — | — |
| 4 | 70.7 | — | — |
| 7 | 73.2 | — | — |

**Answers**

8. *EVSI* increases as $n$ increases.

9. (a) $n = 1$, thus $\frac{1}{S_1^2} = \frac{1}{(.30)^2} + \frac{1}{(.20)^2/1} = 36.1$ or $S_1^2 = .0277$.

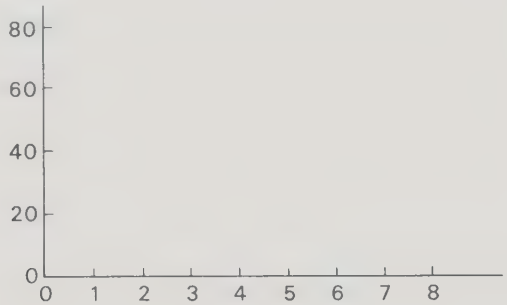
Since $S_0^2 = (.30)^2 = .09$, it follows that:

$S_* = \sqrt{.09 - .0277} = .250$, $D_* = |(1.3 - 1.2)/.250| = .40$, and $L(D_*) = L(.40) = .2304$.

Hence: $EVSI = 1000(.250)(.2304) = \$57.6$ thousand.

(b) $EVSI(n = 1)/EVPI = 57.6/76.65 = .75$. The fact that the expected value of sample information for a single sample observation is 75% of the expected value of perfect information suggests that in this particular problem a small sample provides a great amount of information.

(b) Plot *ENGS* as a function of $n$.√




(c) From your results in (a) and (b), obtain the sample size which yields the maximum expected net gain from sampling._____ What is the amount of this maximum expected net gain? _____

(d) Should sample sizes for which *ENGS* is negative be considered for finding the optimum sample size?

_____√

**F. Selection of Appropriate Act After Sampling.** Once the sample information has been collected, the optimal act is selected as in the no-sample case, except the mean of the posterior probability distribution ($M_1$) is used in the calculation of the expected payoffs. Alternatively, the following equivalent decision rule can be used: If $M_1 \leqslant K$, select that act which is best for $\mu$ values below the indifference point ($K$); if $M_1 > K$, select that act which is best for $\mu$ values above the indifference point.

Example 9 Refer to Example 1. The best act for $\mu$ values below the indifference point, $K = 1.3$, is act $C_1$. For $\mu$ values above $K = 1.3$, act $C_2$ is the best act. Suppose the firm selected a sample of $n = 4$ and found $\bar{X} = 1.3$. Then $M_1 = 1.29$ (see Example 5). Hence act $C_1$ is to be selected.

**Problem 11**

In Example 1, suppose the firm obtained information based on the first nine weeks of operation and found $\bar{X} = 1.4$. On the basis of the sample results given, what act should the firm select? (*Hint*: In this case $M_1 = 1.39$; see Problem 6.)_____√

## REVIEW PROBLEMS

1. Consider Review Problem 1 of Section 29, where a company has a large number of customers for a new product it is developing and it can lease production facilities at either of two locations. Thus, it must decide between: $C_1$: use location one; and $C_2$: use location two. The payoff to the company depends on the mean number of units ($\mu$) which will be sold to each customer during the period of the lease. On the basis of a price and cost analysis, the company has estimated the payoffs (in $ millions) for different values of $\mu$ for the two acts, $C_1$ and $C_2$, to be as follows:

$$P_1 = 1.05 + .05\mu$$
$$P_2 = -2.75 + .25\mu.$$

It is assumed that the distribution of the number of units that will be sold to each customer is normal with standard deviation $\sigma = 16$. Further, the company's management has assigned a prior probability distribution to $\mu$ which is normal with mean $M_0 = 22$ and standard deviation $S_0 = 9$.

(a) What is the value of the indifference point $K$? _____

(b) If the company must make a decision with no sample information, what is the optimal act? _____

(c) What is the absolute value of the slope $t$ of the opportunity loss functions? (*Hint*: It is the absolute difference between the two slopes of the payoff functions.) $t =$ _____ √

(d) Find: $EVPI =$ _____

(e) If the cost of sampling is $c = \$50$ per observation (i.e., $.00005 million), what is the maximum sample size to be considered? _____

(f) What is $EVSI$ if the sample size is $n = 100$, advance orders for these 100 customers are obtained, and the sample mean $\bar{X}$ is used to revise the prior probabilities? $EVSI =$ _____

(g) Is the $EVSI$ for $n = 100$ a large proportion of $EVPI$? _____ What is the significance of this? _____ √

(h) Suppose the company decides to select a sample of $n = 150$ customers and finds $\bar{X} = 17.92$.

(i) Derive the posterior probability distribution. $M_1 =$ _____ ; $S_1 =$ _____

(ii) State the decision rule to be employed. _____

(iii) Which act is optimal? _____

(iv) From the posterior distribution described in (i), determine the probability that $\mu$ exceeds 19. _____ √

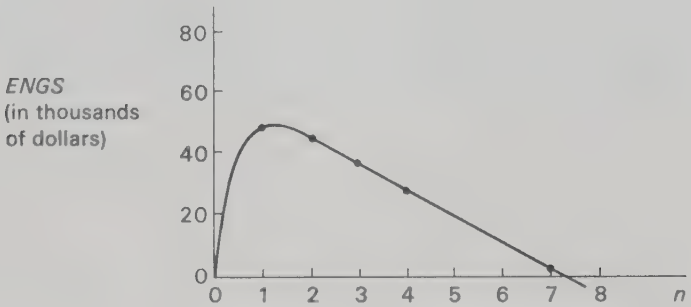
2. If a decision-maker's prior knowledge about the value of $\mu$ is particularly vague, he may wish to use a prior standard deviation, $S_0$, which is very large. What effect would this have on the mean and standard deviation of the posterior distribution? _____ √

Answers

10. (a)

| $n$ | $cn$ | $ENGS$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 10 | 47.6 |
| 2 | 20 | 45.4 |
| 3 | 30 | 38.5 |
| 4 | 40 | 30.7 |
| 7 | 70 | 3.2 |

(b)




(c) Since $n$ must be a whole number, it is seen that a sample size $n = 1$ will yield the maximum expected gain from sampling. For $n = 1$, $ENGS = \$47.6$ thousand.

(d) Since $ENGS = 0$ for $n = 0$, there is no need to consider any sample sizes for which $ENGS$ is negative. For such sample sizes, the expected net payoff would be smaller than that for $n = 0$.

11. Since $K = 1.3$, the decision rule is: If $M_1 \leqslant 1.3$, select act $C_1$; if $M_1 > 1.3$, select act $C_2$. Here $M_1 = 1.39 > 1.3$, thus select act $C_2$.

## ANSWERS TO REVIEW PROBLEMS

1. (a) $K$ is the value of $\mu$ where $P_1 = P_2$, that is, $1.05 + .05K = -2.75 + .25K$. Solving this equation for $K$ gives $K = 19$.

(b) The best act for $\mu$ values above $K = 19$ is $C_2$ and below $K = 19$ is $C_1$. Thus, the decision rule is: If $M_0 \leqslant 19$, select act $C_1$; if $M_0 > 19$, select act $C_2$. Here $M_0 = 22 > 19$, thus, $C_2$ is the optimal act.

(c) $t = |.25 - .05| = .20$ ($ millions per unit per customer).

(d) Here $t = .20$, $S_0 = 9$, $K = 19$, $M_0 = 22$, $D = |(19 - 22)/9| = .33$, and $L(D) = L(.33) = .2555$. Hence: $EVPI = .20(9)(.2555) = \$.460$ million.

(e) $EVPI = .460$ and $c = .00005$. Since $n = 9200$ is the largest value of $n$ for which $cn \leqslant EVPI$, that is, $(.00005)9200 \leqslant .460$, it follows that it is the maximum sample size to be considered.

(f) $n = 100$, thus $\dfrac{1}{S_1^2} = \dfrac{1}{9^2} + \dfrac{1}{(16)^2/100} = .403$ or $S_1^2 = 2.48$.

Since $S_0^2 = 9^2 = 81$, it follows that:

$S_* = \sqrt{81 - 2.48} = 8.86$, $D_* = |(19 - 22)/8.86| = .34$, and $L(D_*) = L(.34) = .2518$.
Hence: $EVSI = (.20)(8.86)(.2518) = \$.446$ million.

(g) $EVSI(n = 100)/EVPI = .446/.460 = .97$. Since $EVSI$ for a sample of 100 is a large proportion of $EVPI$, this indicates that not too much additional value can be expected from substantially larger sample sizes.

## ANSWERS TO REVIEW PROBLEMS

1. cont'd.

(h) (i)

$$M_1 = \frac{\frac{22}{9^2} + \frac{17.92}{(16)^2/150}}{\frac{1}{9^2} + \frac{1}{(16)^2/150}} = 18.00,$$

$$\frac{1}{S_1^2} = \frac{1}{9^2} + \frac{1}{(16)^2/150} = .598 \text{ or } S_1 = 1.29.$$

Thus, the posterior distribution for $\mu$ is normal with mean 18.00 and standard deviation 1.29.

(ii) Since $K = 19$, the decision rule is: If $M_1 \leqslant 19$, select act $C_1$; if $M_1 > 19$, select act $C_2$.

(iii) Since $M_1 = 18.00 \leqslant 19$, select act $C_1$.

(iv) Since the posterior distribution is normal with $M_1 = 18.00$ and $S_1 = 1.29$, it follows that:

$$P(\mu > 19) = P\left(z > \frac{19 - 18.00}{1.29}\right) = P(z > .78) = .2177.$$

2. If $S_0$ has a very large value, from the formulas for $M_1$ and $S_1$ it is seen that $M_1$ will be approximately equal to $\bar{X}$ and $S_1$ will be approximately equal to $\sigma/\sqrt{n}$. Thus, with vague prior knowledge, the characteristics of the posterior distribution are determined almost entirely by the sample information.

# 32

# Index Numbers

An index number is a ratio with a base of 100. In this section, we consider index numbers that summarize changes in prices and quantities.

**A. Weighted Aggregate Price Index.** The weighted aggregate price index for time period $n$ on base period 0 is given by:

$$P_n = \frac{\Sigma p_n q_t}{\Sigma p_0 q_t} \cdot 100.$$

Here $P_n$ denotes the price index for the $n$th period, $p_n$ and $p_0$ denote the unit prices of items in the nth and base periods respectively, and $q_t$ is the typical number of units of the item which are produced, consumed, or the like. The quantities ($q_t$) are called *quantity weights.*

Example 1 Let us construct a weighted aggregate grain price index to measure changes in prices received by farmers in a region for grains during 1966, 1967, and 1968. The average annual production in the region of the relevant grains during the period 1961 to 1965 will be used as the weights, and 1966 is to be the base year. The weighted aggregate price indexes are calculated in Table 32.1.

**Table 32.1**

| Grain | Weights ($q_t$) (billions of bushels) | Prices ($ per bushel) | | | | | |
|---|---|---|---|---|---|---|---|
| | | $p_0$ (1966) | $p_1$ (1967) | $p_2$ (1968) | $p_0q_t$ | $p_1q_t$ | $p_2q_t$ |
| Corn | 3.40 | 1.14 | 1.08 | 1.20 | 3.88 | 3.67 | _____ |
| Wheat | 1.20 | 1.37 | 1.33 | 1.55 | 1.64 | 1.60 | _____ |
| Oats | 1.20 | .64 | .74 | .64 | .77 | .89 | _____ |
| Barley | .40 | .93 | 1.00 | 1.22 | .37 | .40 | _____ |
| Total | | | | | 6.66 | 6.56 | _____ |

Thus we have: $\Sigma p_0 q_t = 6.66$ and $\Sigma p_1 q_t = 6.56$. The price index for the base year, 1966, is: $P_0 = (6.66/6.66)100 = 100$. For 1967, the price index is: $P_1 = (6.56/6.66)100 = 98$. Note that the index for the base period always is 100.

**Problem 1**

Refer to Example 1.

(a) Calculate the weighted aggregate price index for 1968. _____

(b) By what per cent did prices of the four grains increase from 1966 to 1968? _____ Does this indicate that aggregate farm income from the sale of these grains was this much higher in 1968 than in 1966?

_____

_____ √

**Answers**

1. (a)

| $p_2q_t$ |
|---|
| 4.08 |
| 1.86 |
| .77 |
| .49 |
| $\Sigma p_2q_t = 7.20$ |

Hence $P_2 = (7.20/6.66)100 = 108$.

**(b) 8%. It does not necessarily follow that aggregate farm income from the sale of these grains was 8 per cent higher in 1968 than in 1966, since actual quantities sold of the four grains may have differed between these years.**

**Problem 2**

Refer to Example 1.

(a) Calculate the index for 1968, using $q_t = 1$ as the weight for each grain, with 1966 as the base year. _____

(b) Consider the weights used in the price index of Example 1 and the weights used in (a) above. Which set of weights gives more appropriate consideration to the relative importance of the different grains? _____ _____ √

**B. Weighted Relative Price Index.** The weighted relative price index for time period $n$ on base period 0 is given by:

$$P_n = \frac{\Sigma(p_n/p_0)(p_tq_t)}{\Sigma p_tq_t} \cdot 100.$$

The weights in the index, $p_tq_t$, are the typical values of the items which are produced, consumed, or the like, and are called *value weights.* Recall that value = price x quantity.

Example 2 Refer to Example 1. Suppose the average values of the various grain crops for the period 1961–1965 are to be used as the weights in the index. These value weights are shown in Table 32.2. The calculation of the weighted relative price index, with 1966 as the base, proceeds as follows:

**Table 32.2**

| Grain | Value Weights $(p_tq_t)$ ($ billions) | Price Relatives 1966 $p_0/p_0$ | 1967 $p_1/p_0$ | 1968 $p_2/p_0$ | $(p_0/p_0)(p_tq_t)$ | $(p_1/p_0)(p_tq_t)$ | $(p_2/p_0)(p_tq_t)$ |
|---|---|---|---|---|---|---|---|
| Corn | 3.80 | 1.00 | .95 | _____ | 3.80 | 3.61 | _____ |
| Wheat | 2.10 | 1.00 | .97 | _____ | 2.10 | 2.04 | _____ |
| Oats | 1.10 | 1.00 | 1.16 | _____ | 1.10 | 1.28 | _____ |
| Barley | .40 | 1.00 | 1.08 | _____ | .40 | .43 | _____ |
| Total | 7.40 | | | | 7.40 | 7.36 | _____ |

To illustrate the calculation of the price relatives, consider the price relative $p_1/p_0$ for corn. Since $p_1 = 1.08$ and $p_0 = 1.14$ for corn, it follows that $p_1/p_0 = 1.08/1.14 = .95$.

From Table 32.2, we see that $\Sigma(p_0/p_0)(p_tq_t) = 7.40$, $\Sigma(p_1/p_0)(p_tq_t) = 7.36$, and $\Sigma p_tq_t = 7.40$. Hence the price relative index for 1966 is: $P_0 = (7.40/7.40)100 = 100$. Similarly, the price index for 1967 is: $P_1 = (7.36/7.40)100 = 99$.

**Problem 3**

(a) From the data given in Example 1 and Example 2, calculate the weighted relative price index for 1968. _____

(b) Are the weighted aggregate price indexes calculated in Example 1 equal to the weighted relative price indexes calculated in Example 2? _____

(c) Between 1966 and 1968, which of the four grains experienced the greatest relative price increase? _____√

**C. Weighted Aggregate Quantity Index.** The weighted aggregate quantity index for time period $n$ on base period 0 is given by:

$$Q_n = \frac{\Sigma q_n p_t}{\Sigma q_0 p_t} \cdot 100.$$

Here $Q_n$ denotes the quantity index for the $n$th period, and $q_n$ and $q_0$ are the quantities of the item produced, consumed, or the like, during the $n$th and base periods respectively. The weights for the index are the typical unit prices of the items ($p_t$).

Example 3 A firm sells three models of a generator, lightweight, standard, and heavy-duty. The firm wishes to construct a weighted aggregate quantity index of the production of generators for the years 1966, 1967, and 1968. The 1967 unit prices will be used as weights, and 1966 is to be the base year. The weighted aggregate quantity indexes are calculated as follows:

**Table 32.3**

| Model | Weights ($p_t$) (1967 unit prices in $) | Quantities Produced | | | | | |
|---|---|---|---|---|---|---|---|
| | | $q_0$ (1966) | $q_1$ (1967) | $q_2$ (1968) | $q_0p_t$ | $q_1p_t$ | $q_2p_t$ |
| Lightweight | 8.0 | 475 | 422 | 422 | 3800 | ____ | 3376 |
| Standard | 12.5 | 320 | 355 | 390 | 4000 | ____ | 4875 |
| Heavy-duty | 20.0 | 114 | 214 | 276 | 2280 | ____ | 5520 |
| Total | | | | | 10,080 | ____ | 13,771 |

We have $\Sigma q_0 p_t = 10{,}080$ and $\Sigma q_2 p_t = 13{,}771$. Hence, the index for the base year, 1966, is: $Q_0 = (10{,}080/10{,}080)100 = 100$. For 1968, the quantity index is: $Q_2 = (13{,}771/10{,}080)100 = 137$.

**Problem 4**

Refer to Example 3. Calculate the weighted aggregate quantity index for 1967. $Q_1$ = _____√

**D. Weighted Relative Quantity Index.** The weighted relative quantity index for time period $n$ on base period 0 is given by:

$$Q_n = \frac{\Sigma(q_n/q_0)(q_t p_t)}{\Sigma(q_t p_t)} \cdot 100.$$

The weights for the items in the index, $q_t p_t$, are the typical values of the items.

**Problem 5**

Refer to Example 3. Calculate the weighted relative quantity index of production for 1968, using 1966 as the base, and 1967 values as weights. Use Table 32.4 on page 281.

Answers

2. (a) If $q_t = 1$ for each item, the weighted aggregate price index reduces to the following formula:

$$P_n = \frac{\Sigma p_n}{\Sigma p_0} \cdot 100.$$

This index is called the *simple aggregate price index*. For our example, the simple aggregate price index for 1968 is obtained as follows:

| Grain | $q_t$ | $p_0$ | $p_2$ |
|---|---|---|---|
| Corn | 1 | 1.14 | 1.20 |
| Wheat | 1 | 1.37 | 1.55 |
| Oats | 1 | .64 | .64 |
| Barley | 1 | .93 | 1.22 |
| Total | | 4.08 | 4.61 |

$P_2 = (4.61/4.08)100 = 113.$

**(b) The set of weights in Example 1 is usually more appropriate since it takes into consideration the relative importance of the different grains. On the other hand, the weights used in (a) give price changes for each grain equal weight.**

3. (a)

| Grain | $p_2/p_0$ | $(p_2/p_0)(p_tq_t)$ |
|---|---|---|
| Corn | 1.05 | 3.99 |
| Wheat | 1.13 | 2.37 |
| Oats | 1.00 | 1.10 |
| Barley | 1.31 | .52 |
| Total | | 7.98 |

$P_2 = (7.98/7.40)100 = 108.$

(b)

| | $P_0$ | $P_1$ | $P_2$ |
|---|---|---|---|
| Weighted aggregate price index | 100 | 98 | 108 |
| Weighted relative price index | 100 | 99 | 108 |

The index values are not exactly equal. Inspection of the two index formulas shows that the indexes will be equal only if $p_t = p_0$, that is, only if base period prices are used in the value weights of the weighted relative price index, in which case we have:

$$\frac{\Sigma(p_n/p_0)(p_0q_t)}{\Sigma(p_0q_t)} = \frac{\Sigma p_n q_t}{\Sigma p_0 q_t}.$$

**(c) From the price relatives calculated in (a), we see that the price of barley experienced the greatest relative increase, 31 per cent.**

4.

| Model | $q_1p_t$ |
|---|---|
| Lightweight | 3376 |
| Standard | 4438 |
| Heavy-duty | 4280 |
| Total | 12,094 |

$Q_1 = (12{,}094/10{,}080)100 = 120.$

**Table 32.4**

| Model | Value Weights $q_t p_t$ | Quantity Relative $q_2/q_0$ (1968) | $(q_2/q_0)(q_t p_t)$ |
|---|---|---|---|
| Lightweight | 422 x 8 = 3376 | .89 | 3005 |
| Standard | 4438 | 1.22 | 5414 |
| Heavy-duty | | | |
| Total | | | |
| | | | $Q_2$ = |

**E. Fisher's Ideal Index.** When measuring changes in prices or quantities between base period 0 and time period $n$, an index which has several desirable properties is Fisher's ideal index. Fisher's ideal *price* index for period $n$ is given by:

$$P_n = \sqrt{\frac{\Sigma p_n q_0}{\Sigma p_0 q_0} \cdot \frac{\Sigma p_n q_n}{\Sigma p_0 q_n}} \cdot 100.$$

Here $p_n$ and $q_n$ are prices and quantities in time period $n$, and $p_0$ and $q_0$ are prices and quantities in the base period 0.

Fisher's ideal *quantity* index for time period $n$ is given by:

$$Q_n = \sqrt{\frac{\Sigma q_n p_0}{\Sigma q_0 p_0} \cdot \frac{\Sigma q_n p_n}{\Sigma q_0 p_n}} \cdot 100.$$

Example 4 The data in Table 32.5 on the production of fresh and frozen packaged fish is available for two successive years, the base year (0) and the following year (1).

**Table 32.5**

| Type of Fish | Production (millions of pounds) | | Price ($) | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $q_0$ | $q_1$ | $p_0$ | $p_1$ | $q_0p_0$ | $q_1p_0$ | $q_0p_1$ | $q_1p_1$ |
| Cod | 9 | 10 | .33 | .30 | 3.0 | 3.3 | 2.7 | 3.0 |
| Flounder | 28 | 49 | .39 | .35 | 10.9 | 19.1 | 9.8 | 17.2 |
| Haddock | 33 | 37 | .33 | .38 | 11.0 | 12.2 | 12.5 | 14.1 |
| Ocean Perch | 44 | 24 | .23 | .25 | 10.1 | 5.5 | 11.0 | 6.0 |
| Whiting | 3 | 2 | .33 | .20 | 1.0 | .7 | .6 | .4 |
| Other | 35 | 39 | .37 | .41 | 13.0 | 14.4 | 14.4 | 16.0 |
| Total | 152 | 161 | | | 49.0 | 55.2 | 51.0 | 56.7 |

A fish packing company wishes to compute Fisher's ideal price index for period 1. The computation is:

$$P_1 = \sqrt{\frac{51.0}{49.0} \cdot \frac{56.7}{55.2}} \cdot 100 = 103.$$

**Problem 6**

In Example 4, it was found that fish prices, on the average, have increased three per cent over the period. Is this consistent with the fact that several types of fish experienced price decreases?

Answers

5.

| Model | $q_t p_t$ | $q_2/q_0$ | $(q_2/q_0)(q_t p_t)$ |
|---|---|---|---|
| Heavy-duty | 4280 | 2.42 | 10,358 |
| Total | 12,094 | | 18,777 |

$Q_2 = (18{,}777/12{,}094)100 = 155.$

6. Although several types of fish experienced price decreases, others experienced price increases. The weights for the different types of fish were such that on the whole the price increases were given more weight than the price decreases.

**Problem 7**

(a) The fish packing company in Example 4 wishes to study overall production trends for the industry. Calculate Fisher's ideal quantity index using the data given in Table 32.5. $Q_1$ = _____

(b) Does the index value obtained in (a) indicate that, on the average, fish production increased or decreased? _____ By what per cent? _____

(c) The basic production data in Table 32.5 indicate that 152 million pounds of fish were produced in the base year and 161 million pounds were produced in the following year. This is an increase of about six per cent in poundage of fish produced. Is the quantity index obtained in (a) therefore incorrect? _____

_____ √

**Problem 8**

(a) Multiply the decimal equivalent of the ideal price index, $P_1$, obtained in Example 4 by the decimal equivalent of the ideal quantity index, $Q_1$, obtained in Problem 7(a). $(P_1/100)\cdot(Q_1/100)$ = _____

(b) Find the ratio of the total *value* of fish production in period one to the total *value* of fish production in the base period. $\Sigma p_1 q_1/\Sigma p_0 q_0$ = _____ √

(c) Except for a possible rounding error, how do the two values obtained in (a) and (b) compare with one another? _____ What, therefore, is an important property of Fisher's ideal price and quantity indexes? _____

_____ √

**F. Chain Index.** A chain index is an index in which each index number has the previous period as the base of comparison. Hence a chain index consists of a series of link relatives.

Example 5 Values of a weighted aggregate price index for medical care, for the years 1965–1968 inclusive, are shown in Table 32.6. Also shown are the sums of weighted prices for each of the years.

**Table 32.6**

| $n$ | Year | Sum of Weighted Prices ($\Sigma p_n q_t$) | Weighted Aggregate Price Index ($\Sigma p_n q_t/\Sigma p_0 q_t$)100 | Chain Index ($\Sigma p_n q_t/\Sigma p_{n-1} q_t$)100 |
|---|---|---|---|---|
| 0 | 1965 | 38.44 | 100 | 100 |
| 1 | 1966 | 41.63 | 108 | 108 |
| 2 | 1967 | 45.95 | 120 | 110 |
| 3 | 1968 | 47.07 | 122 | ___ |

A chain index for medical care prices is constructed as follows: chain index for 1966 = (41.63/38.44)100 = 108; chain index for 1967 = (45.95/41.63)100 = 110; and so on. In general, the chain index for period $n$ would be given by the formula $(\Sigma p_n q_t/\Sigma p_{n-1} q_t)100$. The index for the first period is arbitrarily taken to be 100.

**Problem 9**

Refer to Example 5.

(a) Compute the chain index value for 1968.

(b) What information does the chain index provide which is not readily available from the weighted aggregate price index? _____ √

(c) Multiply the chain indexes for 1966 and 1967 (in decimal form) by each other, and then multiply the product by the chain index for 1968. Compare the result with the weighted aggregate price index for 1968. Except for rounding how do these two numbers compare? _____ What is the significance of this? _____ √

**G. Adjusting into Constant Dollars.** Various situations arise in which it is useful to express dollar data in terms of constant dollars.

Example 6 A family's food expenditures during 1964–1968 inclusive are shown in Table 32.7 in current dollars. An index series for food prices with base year 1958 also is shown. Assume it is desired to express the expenditures in constant 1958 dollars.

**Table 32.7**

| Year | Food Price Index (1958 = 100) | Annual Food Expenditures | |
|---|---|---|---|
| | | Current Dollars | 1958 Dollars |
| 1964 | 106 | 1219 | ____ |
| 1965 | 110 | 1452 | ____ |
| 1966 | 114 | 1530 | 1342 |
| 1967 | 119 | 1711 | 1438 |
| 1968 | 122 | 1806 | 1480 |

To express the expenditure amounts in constant 1958 dollars, we divide the current dollar amounts by the decimal equivalent of the corresponding index value. For instance, the 1966 expenditure of \$1530 would become 1530/1.14 = \$1342 (1958 dollars).

**Problem 10**

(a) Complete Table 32.7.

(b) By what per cent has the value of food purchases in 1958 dollars increased between 1964 and 1968? _____ √

**H. Shifting Base Periods.** Suppose we had wished to express the annual food expenditures in Example 6 in 1968 dollars. This would require a shift in the base of the index series from 1958 to 1968. This shift is accomplished by dividing each value in the 1958-based index series by the decimal equivalent of the index for 1968, namely 1.22. The division yields a 1968 index series with a value of 122/1.22 = 100 for 1968, 119/1.22 = 98 for 1967, and so on.

**Problem 11**

Refer to Example 6.

(a) Obtain the food price index for 1964, with 1968 as the base. _____

(b) Express 1964 and 1968 food expenditures in constant 1968 dollars. _____ _____ √

(c) By what per cent has the value of food purchases in 1968 dollars increased between 1964 and 1968? _____

(d) How does the result in (c) compare, except for rounding effects, with that in Problem 10 (b)? _____ What generalization is suggested by this? _____ √

Answers

7. (a) $Q_1 = \sqrt{\frac{55.2}{49.0} \cdot \frac{56.7}{51.0}} \cdot 100 = 112.$

(b) The index in (a) indicates that fish production has increased 12 per cent over the one-year period.

(c) No, increases of some types of fish production are more important than others; this is recognized in the index but not in the total weights of fish production.

8. (a) (103/100) (112/100) = 1.154.

(b) From the table in Example 4, we obtain $\Sigma p_1 q_1 / \Sigma p_0 q_0 = 56.7/49.0 = 1.157$.

(c) Except for a rounding error, the two values are equal. For Fisher's ideal indexes, $P_1$ and $Q_1$, we have $P_1 Q_1 = \Sigma p_1 q_1 / \Sigma p_0 q_0$. This is called the *factor reversal* property of Fisher's ideal index. Note that this property permits one to separate a change in value ($\Sigma p_1 q_1 / \Sigma p_0 q_0$) into two components, one component measuring price changes ($P_1$) and the other component measuring quantity changes ($Q_1$).

9. (a) (47.07/45.95)100 = 102.

(b) A chain index shows changes from one period to the next one. This information is not readily seen from a weighted aggregate price index where changes always are shown from the fixed base period.

(c) (108/100) (110/100) (102) = 121. Except for a rounding error, the two values are equal. This demonstrates that by multiplying chain indexes, one obtains the weighted aggregate price index.

10. (a)

| Year | Food Price Index (1958 = 100) | Annual Food Expenditures | |
|---|---|---|---|
| | | Current Dollars | 1958 Dollars |
| 1964 | 106 | 1219 | 1150 |
| 1965 | 110 | 1452 | 1320 |

(b) (1480/1150)100 − 100 = 29%.

11. (a) 106/1.22 = 87.

(b) 1964 food expenditure in 1968 dollars = 1219/.87 = $1401;
1968 food expenditure in 1968 dollars = 1806/1.00 = $1806.

(c) (1806/1401)100 − 100 = 29%.

(d) The two percentages are equal. This demonstrates that per cent changes in constant dollars are the same, regardless of the year of the constant dollars.

Example 7 Shifting of the base of an index is also necessary to join or splice two index series which overlap in time. Shown in Table 32.8 are two retail price index series for solid and petroleum fuels. The series on the left was compiled for 1966 and earlier. In 1966, the weights of the index were changed to reflect changing conditions, and a new series began to be calculated on a 1966 base. To splice the two indexes into one continuous series, it is necessary to give them a common value for 1966. In the present case, it was decided to shift the base of the old index to 1966 so that both series would have a value of 100 in that year and could be spliced.

Table 32.8

| Year | Old Index Series 1958 = 100 | New Index Series 1966 = 100 | Spliced Index Series _____ = 100 |
|---|---|---|---|
| 1964 | 104 | | ____ |
| 1965 | 106 | | ____ |
| 1966 | 107 | 100 | 100 |
| 1967 | | 104 | 104 |
| 1968 | | 110 | 110 |

**Problem 12**

Refer to Example 7.

(a) Shift the base of the old series as indicated, and splice the two series. Record the new index values for 1964 and 1965 in the column of the spliced index series.

(b) What is the base period of the spliced series?_____ √

(c) By what per cent did the prices increase between 1964 and 1966 according to the old series?_____ By what per cent did the prices increase during this same period according to the spliced series?_____ What is the implication of your answers?_____

_____ √

## REVIEW PROBLEMS

1. A company wishes to compute a price index for the two products it produces for the 1967–1968 period. Shown in Table 32.9 are data on manhours expended ($q$) and the prices ($p$) for each product during the years 1967–1968.

**Table 32.9**

| Product | Manhours Expended (in thousands) | | Prices | |
|---|---|---|---|---|
| | 1967 | 1968 | 1967 | 1968 |
| *A* | 600 | 750 | 1.50 | 1.80 |
| *B* | 450 | 500 | 1.90 | 2.20 |

(a) Calculate a price index for 1968 by the weighted aggregate method. Use 1967 as your base year and use data on manhours expended in the base year for your weights. _____

(b) For the same data, calculate a price index for 1968 using Fisher's ideal price index formula._____

(c) What was the per cent increase in prices between 1967 and 1968 as indicated by the index calculated in (a)? _____ √

2. The plant and equipment expenditures of a manufacturing firm during 1966–1968 are shown in Table 32.10. Also shown is a price index of manufacturing plant and equipment.

**Table 32.10**

| Year | Capital Expenditures ($ thousands) | Price Index of Manufacturing Plant and Equipment (1960–1962 = 100) |
|---|---|---|
| 1966 | 108 | 126 |
| 1967 | 57 | 133 |
| 1968 | 156 | 140 |

(a) Express the capital expenditures in constant 1968 dollars. 1966:_____ ; 1967:_____ ; 1968:_____

(b) By what per cent did capital expenditures in constant dollars increase between 1966 and 1968?_____ √

Answers

12. (a)

| Year | Spliced Index Series (1966 = 100) |
|---|---|
| 1964 | (104/1.07) = 97 |
| 1965 | (106/1.07) = 99 |

(b) 1966.

(c) Prices increased (107/104)100 − 100 = 3 per cent between 1964 and 1966 according to the old series and (100/97)100 − 100 = 3 per cent between 1964 and 1966 according to the spliced series. This demonstrates that per cent changes are unaffected by shifting the base. Hence a spliced series preserves relative changes.

## ANSWERS TO REVIEW PROBLEMS

1. (a) The subscripts 0 and 1 represent 1967 and 1968 respectively.

| Product | $p_0q_0$ | $p_1q_0$ |
|---|---|---|
| A | 1.50 x 600 = 900 | 1.80 x 600 = 1080 |
| B | 855 | 990 |
| Total | 1755 | 2070 |

Price index = (2070/1755)100 = 118.

(b)

| Product | $p_0q_0$ | $p_1q_0$ | $p_0q_1$ | $p_1q_1$ |
|---|---|---|---|---|
| A | 900 | 1080 | 1125 | 1350 |
| B | 855 | 990 | 950 | 1100 |
| Total | 1755 | 2070 | 2075 | 2450 |

$$\text{Price index} = \sqrt{\frac{2070}{1755} \cdot \frac{2450}{2075}} \cdot 100 = 118.$$

(c) 18%.

2. (a)

| Year | Price Index of Manufacturing Plant and Equipment (1960–62 = 100) | Price Index of Manufacturing Plant and Equipment (1968 = 100) | Capital Expenditures in 1968 Dollars ($ thousands) |
|---|---|---|---|
| 1966 | 126 | 90 | 108/.90 = 120 |
| 1967 | 133 | 95 | 60 |
| 1968 | 140 | 100 | 156 |

(b) (156/120)100 − 100 = 30%.

# 33

# Trends

Trend analysis is a useful tool for long-term forecasting and other business and economic applications. In this section, we consider the construction of straight line and semilogarithmic trend curves.

**A. Straight Line Trend.** When a time series is encountered whose long-term trend pattern is best described by a straight line, the method of least squares can be used for fitting the straight line trend. We will write the equation of the fitted straight line trend as follows:

$$\overline{Y}_X = a + bX.$$

In time series analysis, $X$ represents the time period and $\overline{Y}_X$ represents the trend value for period $X$. The constant $a$ is the value of $\overline{Y}_X$ when $X = 0$, and $b$ is the slope of the trend line. The constants $a$ and $b$ are computed from the time series data using the following least squares formulas:

$$b = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{n\Sigma X^2 - (\Sigma X)(\Sigma X)},$$

$$a = \overline{Y} - b\overline{X}.$$

Here $n$ is the number of time periods for the data to which the trend curve is fitted.

Example 1 Shown in Table 33.1 are data on life insurance purchases from 1962 to 1968 inclusive. Let us fit a straight line trend to these data.

**Table 33.1**

| Year | Time Period $X$ | Life Insurance Purchases ($ billions) $Y$ | $XY$ | $X^2$ | $\overline{Y}_X$ |
|---|---|---|---|---|---|
| 1962 | −3 | 74 | −222 | 9 | 67.2 |
| 1963 | −2 | 79 | −158 | 4 | 77.8 |
| 1964 | −1 | 80 | − 80 | 1 | ____ |
| 1965 | 0 | 90 | 0 | 0 | ____ |
| 1966 | +1 | 105 | 105 | 1 | ____ |
| 1967 | +2 | 142 | 284 | 4 | ____ |
| 1968 | +3 | 122 | 366 | 9 | ____ |
| Total | 0 | 692 | 295 | 28 | |

With time series data, the computational work will be reduced if the values of the $X$ variable (time) are transformed to simpler numbers which sum to zero. Thus, in the case at hand, we have designated 1962 as year −3, 1963 as year −2, and so on. If the series happened to contain an even number of years, say six, the earliest year would be

designated −2.5, and the following years would be −1.5, −.5, +.5, +1.5, and +2.5. When $\Sigma X = 0$, the formulas for $b$ and $a$ simplify to:

$$b = \frac{\Sigma XY}{\Sigma X^2}$$

$$a = \overline{Y}.$$

To obtain the constants for the trend line, we substitute the data from Table 33.1 into the simplified formulas for $a$ and $b$. Hence:

$$b = 295/28 = 10.536.$$

**Problem 1**

Refer to Example 1.

(a) Compute: $a =$ _____

(b) State the fitted trend equation. _____

(c) What is the $X$ value for 1964? _____

(d) What is the trend value for 1964? _____ √

(e) Complete the remaining trend values in Table 33.1.

(f) Plot the trend line on the graph below, and also the seven pairs of $X,Y$ values. Do the actual life insurance purchases fall exactly on the trend line? ____ Should they? _____ √

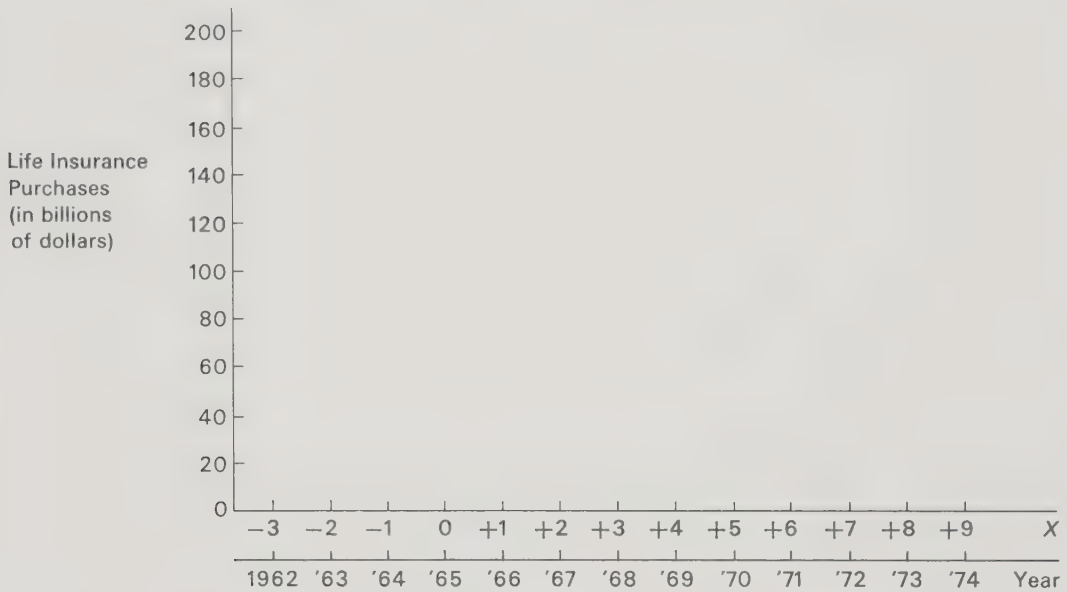



**Figure 33.1**

Example 2 To project the trend curve obtained in Problem 1, one computes the value of $\overline{Y}_X$ for the future value of $X$. The scheme for denoting the $X$ values should be maintained in substituting for $X$ in the trend equation. For example, to compute the trend value of life insurance purchases for, say 1972, one would use $X = +7$ since this is the $X$ value corresponding to 1972 under the system utilized:

$$\overline{Y}_{1972} = 98.857 + 10.536(7) = 172.61.$$

**Problem 2**

(a) Extend the trend line on the graph in Problem 1 to 1972. Is the graphic projection consistent with the calculation in Example 2? _____

(b) What is the trend value for 1974? $\overline{Y}_{1974}$ = _____

(c) Do you expect actual life insurance purchases in 1974 will be equal to the trend value calculated in (b)?

_____

_____ √

**B. Semilogarithmic Trend.** When the long-term pattern of change in a time series is more or less constant in rate, it is appropriate to fit a semilogarithmic trend curve to the data by the method of least squares. This type of trend curve plots as a straight line on a semilogarithmic graph.

The equation of a semilogarithmic trend curve is:

$$\log \overline{Y}_X = a + bX.$$

As before, the constants $a$ and $b$ are calculated from the data using appropriate formulas derived by the method of least squares, as follows:

$$b = \frac{n\Sigma X \log Y - (\Sigma X)(\Sigma \log Y)}{n\Sigma X^2 - (\Sigma X)(\Sigma X)}$$

$$a = \frac{(\Sigma \log Y - b\Sigma X)}{n}.$$

Note that these formulas correspond to those for fitting a straight line trend except that the logarithms of the $Y$ variable are used rather than the actual values of this variable.

When the time variable is expressed so that $\Sigma X = 0$, the formulas for fitting a semilogarithmic trend curve simplify to:

$$b = \frac{\Sigma X \log Y}{\Sigma X^2}$$

$$a = \frac{\Sigma \log Y}{n}.$$

Example 3 The Special Chemicals Division of a large corporation has been experiencing declining sales. The sales data for this division for 1964–1968 inclusive are presented in Table 33.2.

**Table 33.2**

| Year | Time Period $X$ | Sales ($ millions) $Y$ | log $Y$ | $X$ log $Y$ | $X^2$ |
|---|---|---|---|---|---|
| 1964 | −2 | 35.8 | 1.55388 | −3.10776 | 4 |
| 1965 | −1 | 28.7 | 1.45788 | −1.45788 | 1 |
| 1966 | 0 | 23.2 | 1.36549 | 0 | 0 |
| 1967 | +1 | 21.6 | _____ | _____ | 1 |
| 1968 | +2 | 20.6 | _____ | _____ | 4 |
| Total | 0 | | 7.02557 | − .60345 | 10 |

**Problem 3**

(a) Complete Table 33.2.

(b) Compute the constants $a$ and $b$ for a semilogarithmic trend curve. $a$ = _____; $b$ = _____

(c) State the equation of the semilogarithmic trend curve for the sales data. _____ √

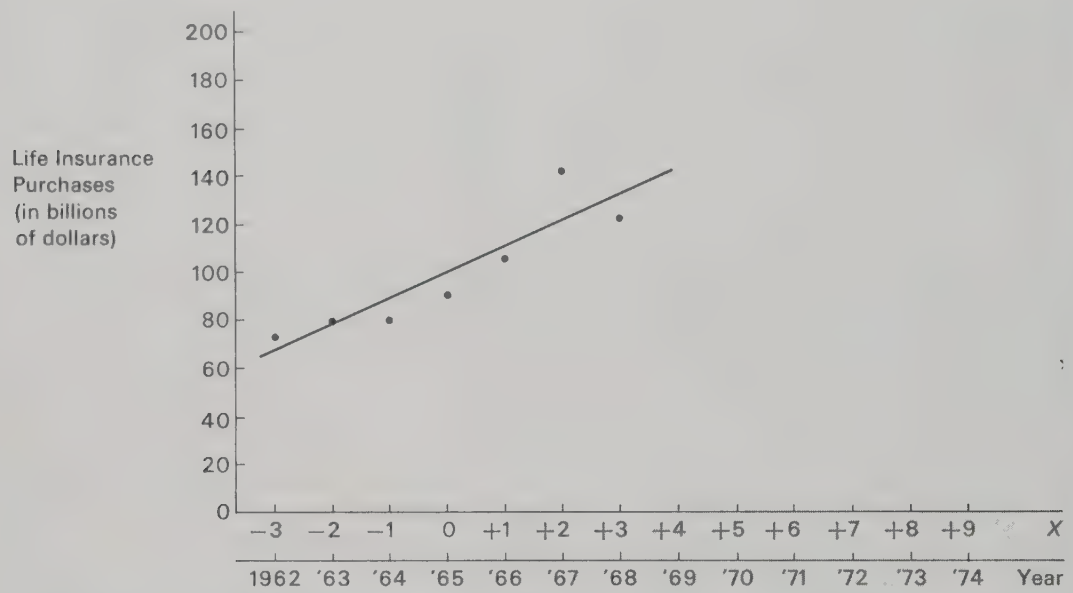
Answers

1. (a) $a = 692/7 = 98.857$.
   (b) $\overline{Y}_X = 98.857 + 10.536X$.
   (c) $X = -1$.
   (d) $\overline{Y}_{1964} = 98.857 + 10.536(-1) = \$88.321$ billion.

   (e)

| Year | $X$ | $\overline{Y}_X$ |
|---|---|---|
| 1964 | −1 | 88.321 |
| 1965 | 0 | 98.857 |
| 1966 | +1 | 109.393 |
| 1967 | +2 | 119.929 |
| 1968 | +3 | 130.465 |

   (f)




   No. The actual values will seldom fall exactly on the trend line because they are affected by cyclical and random factors as well as the trend factor.

2. (a) If the trend line is plotted correctly, the indicated trend value for 1972 must be about $172.6 billion.
   (b) $\overline{Y}_{1974} = 98.857 + 10.536(9) = \$193.681$ billion.
   (c) It is unlikely that the actual life insurance purchases in 1974 will equal the trend value in (b). Assuming that the straight line trend equation accurately describes the trend in the series, and assuming that this trend pattern will continue to hold until 1974, the actual amount of life insurance purchases will fluctuate around the trend value for 1974, but generally will not equal it because of a wide variety of cyclical and random factors.

3. (a)

| Year | $X$ | log $Y$ | $X$ log $Y$ |
|---|---|---|---|
| 1967 | +1 | 1.33445 | 1.33445 |
| 1968 | +2 | 1.31387 | 2.62774 |

   (b) $b = -.60345/10 = -.06034$;
   $a = 7.02557/5 = 1.40511$.
   (c) $\log \overline{Y}_X = 1.40511 - .06034X$.

Example 4 To obtain a trend value $\overline{Y}_X$ from a semilogarithmic trend equation, we first compute the value of $\log \overline{Y}_X$ corresponding to the appropriate value of $X$. We then find the antilogarithm of $\log \overline{Y}_X$ to obtain $\overline{Y}_X$. For example, in the case of Example 3, at 1964, $X = -2$; hence:

$$\log \overline{Y}_{1964} = 1.40511 - .06034(-2) = 1.52579,$$

and thus:

$$\overline{Y}_{1964} = \text{antilog } 1.52579 = 33.6.$$

The trend values for other years may be computed in a similar fashion. Some of these trend values are given in Table 33.3.

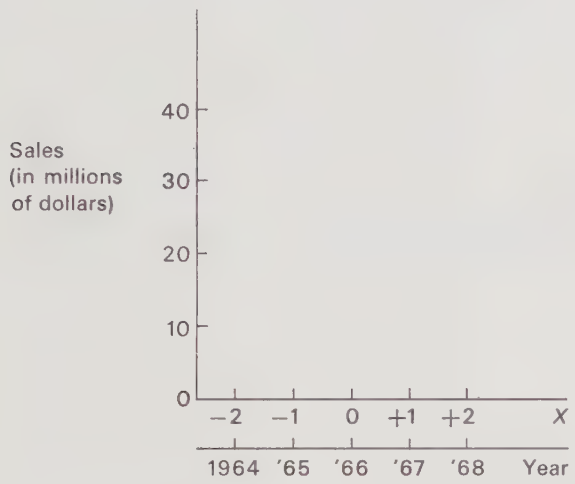
**Table 33.3**

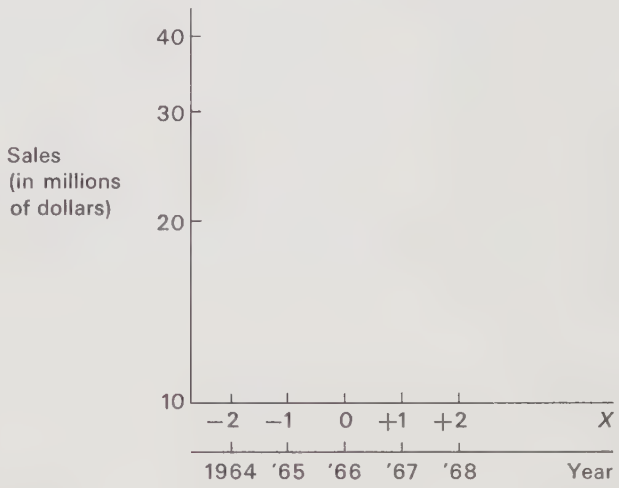
| Year | $X$ | $\log \overline{Y}_X$ | $\overline{Y}_X$ |
|---|---|---|---|
| 1964 | −2 | 1.52579 | 33.6 |
| 1965 | −1 | 1.46545 | 29.2 |
| 1966 | 0 | 1.40511 | 25.4 |
| 1967 | +1 | _____ | ____ |
| 1968 | +2 | _____ | ____ |

**Problem 4**

(a) Complete Table 33.3. √

(b) Plot the five pairs of $X$ and $Y$ values and the trend curve on the arithmetic graph on the left below:




(c) Plot the five pairs of $X$ and $Y$ values and the trend curve on the semilogarithmic graph on the right above.

(d) What does the constant slope of the trend curve in (c) signify? _____

_____ √

Example 5 To project the trend curve obtained in Problem 3, one obtains the value of $\overline{Y}_X$ for the future value of $X$. For example, for the data in Example 3, at 1975, $X = +9$. The projected trend value for 1975 therefore is:

$$\log \overline{Y}_{1975} = 1.40511 - .06034(+9) = .86205,$$

and thus:

$$\overline{Y}_{1975} = \text{antilog } .86205 = 7.3.$$

**Problem 5**

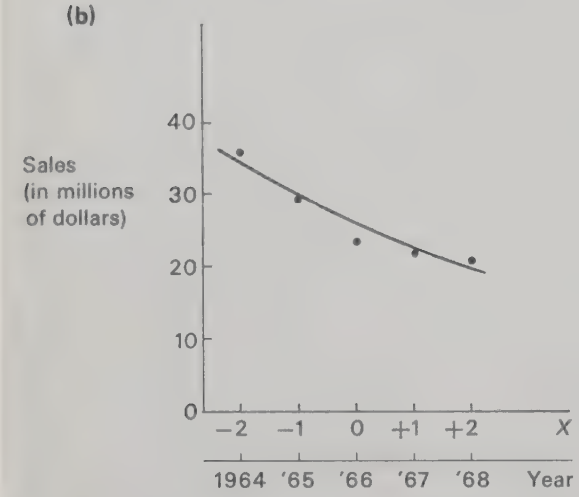
The corporation plans to discontinue operations of the Special Chemicals Division in the year when projected trend sales fall to $10 million. Find this year. _____ √

Answers

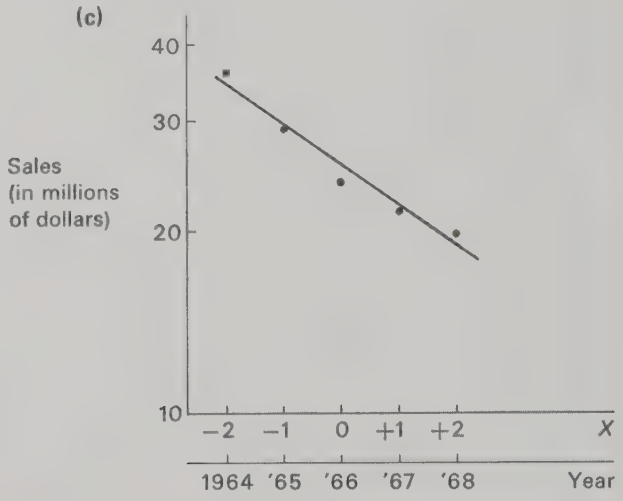
4. (a)

| Year | $X$ | $\log \overline{Y}_X$ | $\overline{Y}_X$ |
|---|---|---|---|
| 1967 | +1 | 1.34477 | 22.1 |
| 1968 | +2 | 1.28443 | 19.2 |

(b)




(c)




(d) **The constant slope signifies that the rate of change in the trend curve is constant.**

5. $\overline{Y}_X = 10.0$. Hence, we wish to solve the following equation for $X$:

$$\log 10.0 = 1.40511 - .06034X.$$

Solving gives $X = 6.7$. The year 1972 corresponds to $X = 6$ and the year 1973 corresponds to $X = 7$. Thus, the company should plan to discontinue operations in the early part of 1973.

Example 6 The constant rate of growth (or decline) implicit in a semilogarithmic trend curve often is measured for descriptive and analytical purposes. This rate of growth ($g$) can be determined from the formula $\log (1 + g) = b$, where $b$ is the slope of the semilogarithmic trend curve.

**Problem 6**

Determine the growth rate $g$ implicit in the trend curve for sales by the Special Chemicals Division (see Problem 3(c)). $g =$ _____ √

**C. Adaptation of Annual Linear Trend Equations to Monthly or Quarterly Basis.** A linear trend equation in the form $\overline{Y}_X = a + bX$ calculated from annual totals, such as annual sales data, can be reduced to a monthly basis by dividing $a$ by 12 and $b$ by 144. The $a$ term then represents the trend value of monthly sales ($\overline{Y}_X$) in the month for which $X = 0$, and $b$ represents the month-to-month increment in the trend of monthly sales. For comparisons with monthly data, trend values should be centered on mid-months. This may be done by adding half of the monthly trend increment to $a$.

A linear trend equation calculated from annual totals is reduced to a quarterly basis by dividing $a$ by 4 and $b$ by 16, and centering the equation appropriately.

Example 7 Consider the trend equation for annual company sales shown on the left in Table 33.4. To reduce this trend equation to a monthly basis, we divide 18 by 12 to obtain 1.5, and divide 28.8 by 144 to obtain .2. The monthly trend equation is shown on the right of the table.

**Table 33.4**

| Annual Trend Equation | Uncentered Monthly Trend Equation |
|---|---|
| $\overline{Y}_X = 18 + 28.8X$ | $\overline{Y}_X = 1.5 + .2X$ |
| $X = 0$ at 1966 | $X = 0$ at July 1, 1966 |
| $X$ in one-year units | $X$ in one-month units |

This monthly trend equation has its origin at July 1, 1966 (if the number of years had been even, the origin would have been at January 1). To center the equation on July 15, 1966, we add .1 (one half of .2) to $a$:

Centered Monthly Trend Equation

$\overline{Y}_X = 1.6 + .2X$
$X = 0$ at July 15, 1966
$X$ in one-month units

**Problem 7**

The trend equation of annual life insurance purchases obtained in Problem 1 was $\overline{Y}_X = 98.857 + 10.536X$. Reduce this trend equation to a monthly basis and center it appropriately. _____ √

## REVIEW PROBLEMS

1. Shown in Table 33.5 are the 1963–1968 year-end credit balances of an oil company. It is desired to construct a trend curve for these data.
   (a) Complete Table 33.5.

**Table 33.5**

| Year | Time Period $X$ | Year-End Credit Balance ($ millions) $Y$ | $XY$ | log $Y$ | $X$ log $Y$ | $X^2$ |
|---|---|---|---|---|---|---|
| 1963 | −2.5 | 47 | −117.5 | 1.67210 | −4.18025 | 6.25 |
| 1964 | −1.5 | 49 | − 73.5 | 1.69020 | −2.53530 | 2.25 |
| 1965 | − .5 | 53 | − 26.5 | 1.72428 | − .86214 | .25 |
| 1966 | + .5 | 59 | + 29.5 | 1.77085 | + .88542 | .25 |
| 1967 | ____ | 68 | ____ | _____ | _____ | ____ |
| 1968 | ____ | 88 | ____ | _____ | _____ | ____ √ |
| Total | 0 | 364 | 134.0 | 10.63442 | .91770 | 17.50 |

   (b) Calculate the equation of the linear trend curve for the credit balance data. _____
   (c) Calculate the equation of the semilogarithmic trend curve for the credit balance data. _____ _____ √
   (d) Based on your semilogarithmic trend curve, what is the approximate annual rate at which the year-end credit balances are growing? _____
   (e) Find the 1972 trend projections for the linear and semilogarithmic trend curves. _____ _____ Why are the projections not identical? _____ _____ √

**Answers**

6. $\log (1 + g) = -.06034$, $1 + g = \text{antilog}(-.06034) = .870$, and thus $g = .870 - 1 = -.130$. The rate of decline in sales is 13.0 per cent per year.

7.

| Annual Trend Equation | Uncentered Monthly Trend Equation | Centered Monthly Trend Equation |
|---|---|---|
| $\overline{Y}_X = 98.857 + 10.536X$ | $\overline{Y}_X = 8.238 + .07317X$ | $\overline{Y}_X = 8.275 + .07317X$ |
| $X = 0$ at 1965 | $X = 0$ at July 1, 1965 | $X = 0$ at July 15, 1965 |
| $X$ in one-year units | $X$ in one-month units | $X$ in one-month units |

## ANSWERS TO REVIEW PROBLEMS

1. (a)

| Year | $X$ | $XY$ | $\log Y$ | $X \log Y$ | $X^2$ |
|---|---|---|---|---|---|
| 1967 | +1.5 | +102 | 1.83251 | +2.74877 | 2.25 |
| 1968 | +2.5 | +220 | 1.94448 | +4.86120 | 6.25 |

(b) $b = 134.0/17.50 = 7.66$; $a = 364/6 = 60.67$;

$$\overline{Y}_X = 60.67 + 7.66X.$$

(c) $b = .91770/17.50 = .05244$; $a = 10.63442/6 = 1.77240$;

$$\log \overline{Y}_X = 1.77240 + .05244X.$$

(d) $\log (1 + g) = .05244$, $1 + g = \text{antilog}(.05244) = 1.13$, and thus $g = .13$. The year-end credit balances are growing at approximately the rate of 13 per cent per year.

(e) $X = 6.5$ corresponds to 1972.
Arithmetic trend projection: $\overline{Y}_{1972} = 60.67 + 7.66(6.5) = \$110$ million.
Semilogarithmic trend projection:

$$\log \overline{Y}_{1972} = 1.77240 + .05244(6.5) = 2.11326;$$
$$\overline{Y}_{1972} = \text{antilog}(2.11326) = \$130 \text{ million}.$$

The two projections differ because they are based on different trend models.

# 34

# Seasonal Variation

Seasonal variation in a time series is a recurrent cyclical pattern of change, where the duration of a full cycle of change is a year or less. In this section, we concern ourselves with yearly seasonal patterns, but the methods discussed are extended readily to other seasonal patterns.

**A. Indexes of Seasonal Variation.** A common method of calculating indexes of seasonal variation is the method of *ratio-to-moving-average*.

Example 1 The first step in the ratio-to-moving-average method is designed to isolate the seasonal and random effects in the time series. To study the logic and procedures of the method, consider the following example.

Given in column 1 of Table 34.1 are quarterly sales data for a department store between 1965 and 1968 inclusive:

**Table 34.1**

| Year and Quarter | (1) Sales ($ thousands) | (2) Four-Quarter Moving Total | (3) Four-Quarter Moving Average |
|---|---|---|---|
| 1965 1 | 2656 | | |
| 2 | 3188 | 12,531 | 3132.75 |
| 3 | 3471 | 13,165 | 3291.25 |
| 4 | 3216 | 13,842 | 3460.50 |
| 1966 1 | 3290 | 14,407 | 3601.75 |
| 2 | 3865 | 14,760 | 3690.00 |
| 3 | 4036 | 15,120 | 3780.00 |
| 4 | 3569 | 15,613 | 3903.25 |
| 1967 1 | 3650 | 16,079 | 4019.75 |
| 2 | 4358 | 16,767 | 4191.75 |
| 3 | 4502 | | |
| 4 | 4257 | ——— | ——— |
| 1968 1 | 4454 | ——— | ——— |
| 2 | 5132 | ——— | ——— |
| 3 | 5379 | ——— | ——— |
| 4 | 4855 | | |

From the data in column 1 we compute a series of four-quarter averages, each of which is to be related to the center of its four-quarter period. First, we compute in column 2 successive four-quarter totals for the data in column 1. For instance, the sales during the four quarters of 1965 totalled $12,531 thousand. The sum is placed

midway in the four-quarter interval, namely midway between the second and third quarters of 1965. The four-quarter total for the period beginning with the second quarter of 1965 and ending with the first quarter of 1966 is $13,165 thousand; this figure is placed midway between the third and fourth quarters of 1965. Successive four-quarter totals can be obtained expeditiously by subtracting, from the latest four-quarter total, the figure for the quarter being dropped in the new total and adding that for the new quarter being picked up. For example, the four-quarter total for the period beginning with the third quarter of 1965 and ending with the second quarter of 1966 is 13,165 minus 3188 (second quarter, 1965) plus 3865 (second quarter, 1966), or $13,842 thousand. This figure is placed midway between the fourth quarter of 1965 and the first quarter of 1966.

In column 3, each of the four-quarter totals is divided by 4 to obtain four-quarter moving averages.

**Problem 1**

Complete columns 2 and 3 of Table 34.1.√

Example 2 Refer to Example 1. To isolate the pattern of seasonal and short-term random variation in the time series, we shall express actual sales during each quarter as a per cent of the average quarterly sales for the four-quarter period centered at that quarter.

Before this can be done, however, an additional step must be taken. Note that the averages in column 3 (restated in Table 34.2) correspond to points of time midway between two quarters, and are thus a half-quarter out of phase with the actual data in column 1. The moving averages are placed in phase by averaging, in groups of two, the successive four-quarter averages of column 3. To do this systematically, successive two-quarter totals of the data in column 3 are placed in column 4 and are converted to quarterly averages in column 5. For example, 6424.00, the first figure in column 4, represents the sum of 3132.75 and 3291.25, and is placed midway between the two figures, corresponding to third quarter 1965. The two-quarter totals then are divided by 2 and the "centered" or in-phase averages entered in column 5.

**Table 34.2**

| Year and Quarter | (1) Sales ($ thousands) | (3) Four-Quarter Moving Average | (4) Two-Quarter Moving Total | (5) Centered Four-Quarter Moving Average | (6) Specific Seasonal Relative |
|---|---|---|---|---|---|
| 1965 1 | 2656 | | | | |
| 2 | 3188 | 3132.75 | | | |
| 3 | 3471 | 3291.25 | 6424.00 | 3212.00 | 108.06 |
| 4 | 3216 | 3460.50 | 6751.75 | 3375.88 | 95.26 |
| 1966 1 | 3290 | 3601.75 | 7062.25 | 3531.12 | 93.17 |
| 2 | 3865 | 3690.00 | 7291.75 | 3645.88 | 106.01 |
| 3 | 4036 | 3780.00 | 7470.00 | 3735.00 | 108.07 |
| 4 | 3569 | 3903.25 | 7683.25 | 3841.62 | 92.90 |
| 1967 1 | 3650 | 4019.75 | 7923.00 | 3961.50 | 92.13 |
| 2 | 4358 | 4191.75 | 8211.50 | 4105.75 | 106.18 |
| 3 | 4502 | 4392.75 | ——— | ——— | ——— |
| 4 | 4257 | 4586.25 | ——— | ——— | ——— |
| 1968 1 | 4454 | 4805.50 | ——— | ——— | ——— |
| 2 | 5132 | 4955.00 | ——— | ——— | ——— |
| 3 | 5379 | | | | |
| 4 | 4855 | | | | |

In column 6, the seasonal and short-term random movements finally are isolated by dividing the actual sales during a quarter (column 1) by the corresponding centered four-quarter moving average (column 5). Each resulting quotient is expressed as a per cent and is called a *specific seasonal relative*. For example, the specific seasonal relative for the third quarter of 1965 is (3471/3212.00)100 = 108.06. Note, incidentally, that the procedure used does not enable us to obtain specific seasonal relatives for the first two or last two quarters of the period.

**Problem 2**

(a) Complete columns 4, 5, and 6 of Table 34.2.
(b) State the meaning of the specific seasonal relative 108.06 for the third quarter of 1965._____
(c) Explain why the specific seasonal relatives for first quarter 1966 and for first quarter 1967 differ.
(d) Explain why the specific seasonal relatives for first quarter 1966 and second quarter 1966 differ. √

Example 3 Refer to Example 2. To obtain the indexes of seasonal variation, it is helpful to assemble the specific seasonal relatives for each quarter into a table like Table 34.3. (NA indicates that a specific seasonal relative is not available for that period.)

**Table 34.3**

| | Quarter | | | | |
|---|---|---|---|---|---|
| Year | 1 | 2 | 3 | 4 | Total |
| 1965 | NA | NA | 108.06 | 95.26 | |
| 1966 | 93.17 | 106.01 | _____ | _____ | |
| 1967 | 92.13 | 106.18 | _____ | _____ | |
| 1968 | 94.85 | 105.16 | NA | NA | |
| Median | 93.17 | 106.01 | _____ | _____ | 402.06 |

Assuming the seasonal pattern is stable over time, this pattern is isolated by averaging the specific seasonal relatives for each quarter. Averaging tends to eliminate random effects by a cancelling of these effects in the average. It is customary to use an average such as the median, which does not give undue weight to extreme specific seasonal relatives. Here we shall use the median of the specific seasonal relatives for each of the four quarters. For example, the median of the three first-quarter specific seasonal relatives is 93.17. This median is referred to as a *typical seasonal relative.*

**Problem 3**

Complete Table 34.3 and obtain the typical seasonal relatives for the third and fourth quarters. √

Example 4 Since a yearly seasonal pattern completes itself, by definition, within a year, the four typical seasonal relatives should average to 100, or equivalently sum to 400. In Example 3, the typical seasonal relatives total 402.06. To adjust them to sum to 400, we multiply each by the ratio 400/402.06. For instance, the adjusted typical seasonal relative for the first quarter is (400/402.06)93.17 = 92.69. This adjusted typical seasonal relative is the *seasonal index* for the first quarter.

**Table 34.4**

| Quarter | 1 | 2 | 3 | 4 | Total |
|---|---|---|---|---|---|
| Typical Seasonal Relative (unadjusted) | 93.17 | 106.01 | 108.06 | 94.82 | 402.06 |
| Seasonal Index | 92.69 | 105.47 | _____ | _____ | 400.00 |

**Problem 4**

(a) Compute the seasonal indexes for the third and fourth quarters in Table 34.4.
(b) Suppose the typical seasonal relatives in Table 34.4 had summed to 397.0. By what factor would you multiply each relative so that the seasonal indexes sum to 400?_____ √

Answers

1.

| Year and Quarter | Four-Quarter Moving Total | Four-Quarter Moving Average |
|---|---|---|
| 1967 3 | 17,571 | 4392.75 |
| 4 | 18,345 | 4586.25 |
| 1968 1 | 19,222 | 4805.50 |
| 2 | 19,820 | 4955.00 |
| 3 | | |

2. (a)

| Year and Quarter | Two-Quarter Moving Total | Centered Four-Quarter Moving Average | Specific Seasonal Relative |
|---|---|---|---|
| 1967 3 | 8584.50 | 4292.25 | 104.89 |
| 4 | 8979.00 | 4489.50 | 94.82 |
| 1968 1 | 9391.75 | 4695.88 | 94.85 |
| 2 | 9760.50 | 4880.25 | 105.16 |

(b) **The specific seasonal relative for the third quarter of 1965 states that actual sales during that quarter were 108.06 per cent as great as average quarterly sales for the four-quarter period centered at that quarter.**

(c) **Given that the seasonal pattern is stable, specific seasonal relatives for the same quarter in different years may differ because of the influence of random, irregular movements in the time series.**

(d) **Differences between specific seasonal relatives of different quarters in the same year result from seasonal and irregular movements in the time series.**

3.

| | Quarter | |
|---|---|---|
| Year | 3 | 4 |
| 1965 | 108.06 | 95.26 |
| 1966 | 108.07 | 92.90 |
| 1967 | 104.89 | 94.82 |
| 1968 | NA | NA |
| Median | 108.06 | 94.82 |

4. (a)

| Quarter | 3 | 4 |
|---|---|---|
| Seasonal Index | 107.51 | 94.33 |

(b) **400/397.0.**

**B. Stable and Moving Seasonal Patterns.** A seasonal pattern in a time series may be fairly stable over time, or it may be shifting from year to year. We can investigate the stability of the seasonal pattern by plotting the specific seasonal relatives on separate graphs for each quarter, and analyzing the movements in the relatives.

Example 5 Refer to the specific seasonal relatives in Table 34.3. The specific seasonal relatives of each quarter are plotted on separate graphs in Figure 34.1.

**Problem 5**

Refer to Example 5.

(a) Plot the specific seasonal relatives for the fourth quarter.

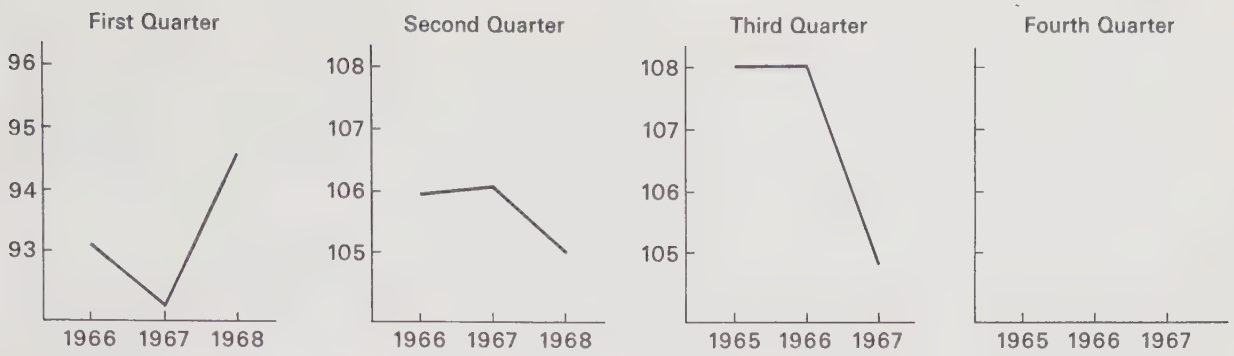



**Figure 34.1**

(b) Study Figure 34.1. Are the specific seasonal relatives for any quarter changing in the same direction from year to year?_____ Does it therefore seem more likely that the seasonal pattern is changing systematically or that the year-to-year movements in each quarter's specific seasonal relatives are due to random, irregular movements in the time series?_____

(c) If the seasonal pattern were changing in a consistent manner, would it be meaningful to compute typical seasonal relatives? _____ √

**Example 6** When a seasonal pattern is changing in a consistent manner, it is called a *moving seasonal pattern.* To measure seasonal variation when the pattern is gradually shifting, we utilize the underlying trend in the shifting pattern to arrive at measures of seasonal variation for each year. In effect, we fit a trend curve to each quarter's specific seasonal relatives. The seasonal index for a given quarter of a particular year is then derived from the ordinate of the trend curve for that year.

For instance, a sales finance company studied the seasonal variation in the volume of its installment financing. Graphs of the quarterly specific seasonal relatives for the years 1963 to 1968 inclusive are shown in Figure 34.2. From analysis of the graphs and knowledge of the factors causing seasonal variation, it was concluded that the seasonal pattern is shifting over time.

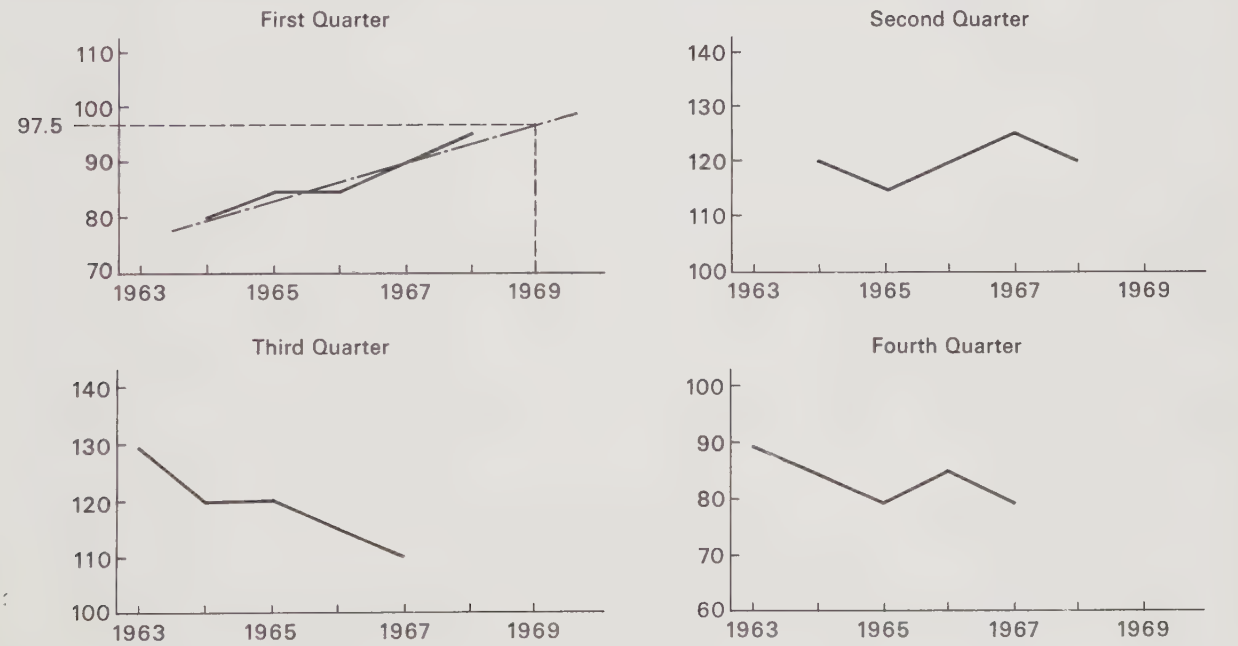
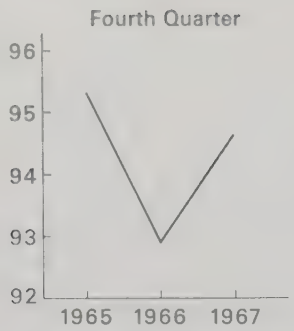



**Figure 34.2**

**Answers**

5. (a)




**(b) No, the directional changes in the specific seasonal relatives are not consistent. Hence, it is likely that the seasonal pattern is stable and that variation in the specific seasonal relatives is due to random, irregular movements in the time series.**

**(c) No, it wouldn't be meaningful to compute typical (average) seasonal relatives. Different seasonal relatives would need to be obtained for each year in this case.**

Suppose the firm wishes to compute a set of seasonal indexes for 1969. To obtain the seasonal index for the first quarter of 1969, we fit a trend line to the first-quarter specific seasonal relatives and project the trend line to 1969. The ordinate of the trend line in 1969 is 97.5. This is the unadjusted seasonal relative for the first quarter of 1969.

**Problem 6**

Refer to Example 6.

(a) For each of the remaining quarters, fit a linear trend by sight to the specific seasonal relatives. For each trend line, estimate the unadjusted seasonal relative for 1969 for that quarter. Quarterly unadjusted seasonal relative: 1st = 97.5; 2nd = _____; 3rd = _____; 4th = _____ √

(b) Construct the quarterly seasonal indexes for 1969 by adjusting the seasonal relatives given in our answer panel for (a) so they total 400. (Two seasonal indexes have been calculated already.) Quarterly seasonal index: 1st = 98.1; 2nd = 123.8; 3rd = _____, 4th = _____ √

**C. Applications of Seasonal Indexes.** Two major applications of seasonal indexes are illustrated below.

**Example 7** Seasonal indexes are used for deseasonalizing or seasonally adjusting time series data. For instance, consider the sales data of Example 1. Quarterly seasonal indexes for these data were 92.69, 105.47, 107.51, and 94.33 for the respective quarters. (See Example 4.) To deseasonalize the sales data, we divide each quarterly sales amount by the decimal equivalent of the appropriate seasonal index. For example, the deseasonalized sales amount for the first quarter of 1968 is $4454 thousand (actual sales during the first quarter of 1968) divided by .9269 (the decimal equivalent of the first-quarter seasonal index) or 4454/.9269 = $4805 thousand. Similarly, the deseasonalized sales amount for the second quarter of 1968 is 5132/1.0547 = $4866 thousand. The deseasonalized sales data for the first two quarters of 1968 are shown in Table 34.5.

**Table 34.5**

| Year and Quarter | Sales | Deseasonalized Sales | Year and Quarter | Sales | Deseasonalized Sales |
|---|---|---|---|---|---|
| 1968 1 | 4454 | 4805 | 1968 3 | 5379 | _____ |
| 2 | 5132 | 4866 | 4 | 4855 | _____ |

**Problem 7**

Refer to Example 7.

(a) Compute the deseasonalized quarterly sales for the third and fourth quarters of 1968.

(b) Interpret the meaning of the deseasonalized quarterly sales of $4805 thousand for the first quarter of 1968. _____ √

(c) Obtain the seasonally adjusted annual rate of sales for the fourth quarter of 1968. _____ _____ Why does this annual rate differ from actual total sales of $19,820 thousand in 1968?

(d) If no changes other than seasonal changes occurred between the fourth quarter of 1968 and the first quarter of 1969, what would be the expected sales for first quarter 1969? _____ √

Example 8 Seasonal indexes also are useful in forecasting. For example, consider the sales finance company of Example 6. In the answer to Problem 6(b), we projected quarterly seasonal indexes of 98.1, 123.8, 101.6, and 76.5 respectively for 1969. Suppose the company forecasts a $48.0 million volume of installment financing in 1969. The seasonal indexes may be used to break down the annual forecast into quarterly forecasts. (To avoid complications, it is assumed here that trend-cyclical influences on installment financing during 1969 are negligible.)

The annual forecast of $48.0 million for 1969 is equivalent to an average quarterly volume of $12.0 million in that year. Since the first-quarter seasonal index for 1969 is 98.1, this indicates that the expected volume of financing in the first quarter will be 98.1 per cent of the average quarterly amount, or 12.0 (.981) = $11.8 million. The forecasts of quarterly financing for the other quarters can be calculated in a similar manner, as shown in Table 34.6.

**Table 34.6**

| Quarter | Average Quarterly Financing ($ millions) | Quarterly Seasonal Index | Forecast of Quarterly Financing ($ millions) |
|---|---|---|---|
| 1 | 12.0 | 98.1 | 12.0 x .981 = 11.8 |
| 2 | 12.0 | 123.8 | 14.9 |
| 3 | 12.0 | 101.6 | ____ |
| 4 | 12.0 | 76.5 | ____ |
| Total | 48.0 | 400.0 | |

**Problem 8**

Refer to Example 8.

(a) Compute the forecasts of quarterly financing for the third and fourth quarters of 1969.

(b) What is the sum of the quarterly forecasts? _____ Except for rounding, does it equal the forecast annual volume? _____

(c) Suppose actual financing volume for the first quarter of 1969 is $10.3 million. Is this volume in line with a $48 million year? _____ √

Answers

6. (a) Because you have fitted your trend lines by sight, your quarterly seasonal indexes may differ from those given below, which were obtained from trend lines fitted by the method of least squares.

Quarterly unadjusted seasonal relatives for 1969:
1st = 97.5; 2nd = 123.0; 3rd = 101.0; 4th = 76.0.

(b) 97.5 + 123.0 + 101.0 + 76.0 = 397.5.
To obtain the quarterly seasonal indexes, multiply each of the unadjusted seasonal relatives in (a) by the ratio 400/397.5, giving:
Quarterly seasonal indexes for 1969:
1st = (400/397.5)97.5 = 98.1; 2nd = 123.8; 3rd = 101.6; 4th = 76.5.

7. (a)

| Year and Quarter | Sales | Deseasonalized Sales |
|---|---|---|
| 1968 3 | 5379 | 5379/1.0751 = 5003 |
| 4 | 4855 | 4855/.9433 = 5147 |

(b) The seasonally adjusted quarterly sales of $4805 thousand represents the average quarterly sales volume that would be expected for the year if no changes except regular seasonal ones occur during the year. Thus, if all factors other than seasonal ones remain constant throughout the year, then average quarterly sales for the year will be $4805 thousand or, in other words, total sales for the year will be 4 x 4805 = $19,220 thousand. The amount $19,220 thousand represents the seasonally adjusted annual rate of first quarter 1968 sales.

(c) 4 x 5147 = $20,588 thousand. The seasonally adjusted annual rate differs from the actual total sales for the year because the seasonally adjusted annual rate only indicates what total sales for the year will be if no changes from the actual sales for the quarter occur other than seasonal changes. If changes other than seasonal changes occur, then the actual sales for the year will not be the same as the seasonally adjusted annual rate for that quarter.

(d) If no changes other than seasonal changes occurred between the fourth quarter of 1968 and the first quarter of 1969, then deseasonalized quarterly sales in both periods would be the same, namely $5147 thousand (see the answer to Problem 7(a)). Since the seasonal index for the first quarter is 92.69, expected sales for first quarter 1969 would be .9269(5147) = $4771 thousand.

8. (a)

| Quarter | Forecast of Quarterly Financing |
|---|---|
| 3 | 12.2 |
| 4 | 9.2 |

(b) 11.8 + 14.9 + 12.2 + 9.2 = 48.1. Except for rounding, the sum equals the forecast annual volume of 48.0.

(c) No, the actual volume of $10.3 million is substantially below the forecast volume of $11.8 million for that quarter, based on a $48 million year.

## REVIEW PROBLEMS

1. Shown in Table 34.7 are quarterly sales (in $ thousands) of a company during 1968, together with a seasonal index pertinent to these sales.

**Table 34.7**

| Quarter | Sales | Seasonal Index | Quarter | Sales | Seasonal Index |
|---|---|---|---|---|---|
| 1 | 3649 | 89 | 3 | 3744 | 104 |
| 2 | 3268 | 86 | 4 | 3993 | 121 |

(a) Remove the influence of seasonal factors to determine the seasonally adjusted pattern of change in sales during 1968. _____ _____ _____ _____

(b) What has been the effect of trend-cyclical factors during the year?_____
_____ √

2. A financial paper recently reported that "inventories of all manufacturers rose to a seasonally adjusted $84.9 billion at the end of July from a seasonally adjusted $84.6 billion a month earlier."

(a) Does this mean that actual inventories of manufacturers amounted to $84.6 billion in June?
_____
_____

(b) Does this mean that the actual inventory levels increased between June and July? _____
_____
_____ √

## ANSWERS TO REVIEW PROBLEMS

1. (a)

| Quarter | Deseasonalized Sales | Quarter | Deseasonalized Sales |
|---|---|---|---|
| 1 | 3649/.89 = 4100 | 3 | 3600 |
| 2 | 3800 | 4 | 3300 |

(b) The seasonally adjusted sales data indicate that the underlying trend-cyclical movement was steadily downward during the year.

2. (a) In general, seasonally adjusted inventory amounts do not represent the actual inventory levels. Rather, they refer to the average inventory level for the year which would result if the level for the period under consideration were changed throughout the year according to the regular seasonal pattern only.

(b) Since actual inventory levels cannot be inferred from seasonally adjusted data without knowledge of the seasonal index, one cannot tell what changes in inventory levels actually occurred. Indeed, there may have been an actual decrease, as illustrated below, even though seasonally adjusted inventories rose.

| Month | Actual Inventories | Seasonal Index | Seasonally Adjusted Inventories |
|---|---|---|---|
| June | 88.0 | 104.0 | 84.6 |
| July | 86.6 | 102.0 | 84.9 |

# 35

# Cyclical Fluctuations

The classical method of time series analysis employs a model in which the current level of a time series ($Y$) is viewed as a function of four components. These components are: trend ($T$), cyclical ($C$), seasonal ($S$), and irregular ($I$) respectively. The model we consider is:

$$Y = T \cdot C \cdot S \cdot I.$$

This model is called the multiplicative classical model.

The trend component is isolated by fitting a trend curve, and is measured in terms of absolute magnitudes. The fitting of two types of trend curves was discussed in Section 33. The other components are measured in terms of ratios, usually per cents. In Section 34, we measured the seasonal component by a method that gave seasonal indexes as per cents that averaged to 100. In this section, we isolate the cyclical-random component by taking per cents of trend using deseasonalized data.

**A. Deseasonalizing Data.** As a first step in isolating the cyclical-random component of a time series, it is necessary to deseasonalize the data whenever the time period covered by each observation is less than one year.

Example 1 A milling company wishes to isolate the cyclical-random component of quarterly sales data for the period 1933–1968 inclusive. The quarterly seasonal pattern has changed over the thirty-six year period. An investigation indicates that the seasonal indexes in Table 35.1 are appropriate.

**Table 35.1**

| | Seasonal Index | |
|---|---|---|
| | For 1933–1947 | For 1948–1968 |
| Quarter | Inclusive | Inclusive |
| 1 | 110 | 108 |
| 2 | 104 | 104 |
| 3 | 86 | 90 |
| 4 | 100 | 98 |

To deseasonalize the quarterly sales data we proceed, as described in Topic C, Section 34, by dividing each quarterly sales amount by the decimal equivalent of the appropriate seasonal index. Some of the deseasonalized sales data are presented in Table 35.2 on page 306. Most of the data have been omitted for conciseness.

**Problem 1**

Refer to Example 1. Deseasonalize the sales data for the first and second quarters of 1934 and the third and fourth quarters of 1968 in Table 35.2.

Table 35.2

| Year and Quarter | | Sales ($ millions) | Deseasonalized Sales ($ millions) |
|---|---|---|---|
| 1933 | 1 | 22.8 | 20.7 |
| | 2 | 21.6 | 20.8 |
| | 3 | 18.1 | 21.0 |
| | 4 | 21.4 | 21.4 |
| 1934 | 1 | 26.4 | —— |
| | 2 | 27.8 | —— |
| | . | . | . |
| | . | . | . |
| | . | . | . |
| 1967 | 3 | 115.3 | 128.1 |
| | 4 | 128.8 | 131.4 |
| 1968 | 1 | 141.7 | 131.2 |
| | 2 | 137.1 | 131.8 |
| | 3 | 118.7 | —— |
| | 4 | 130.2 | ——√ |

**B. Per Cent of Trend.** The second step in isolating the cyclical-random component of a time series is to adjust the deseasonalized data for trend. This may be accomplished by fitting a trend curve to the deseasonalized data and calculating the ratio of the deseasonalized value to the trend value, for each time period. This ratio, expressed as a per cent, is referred to as a *per cent of trend*.

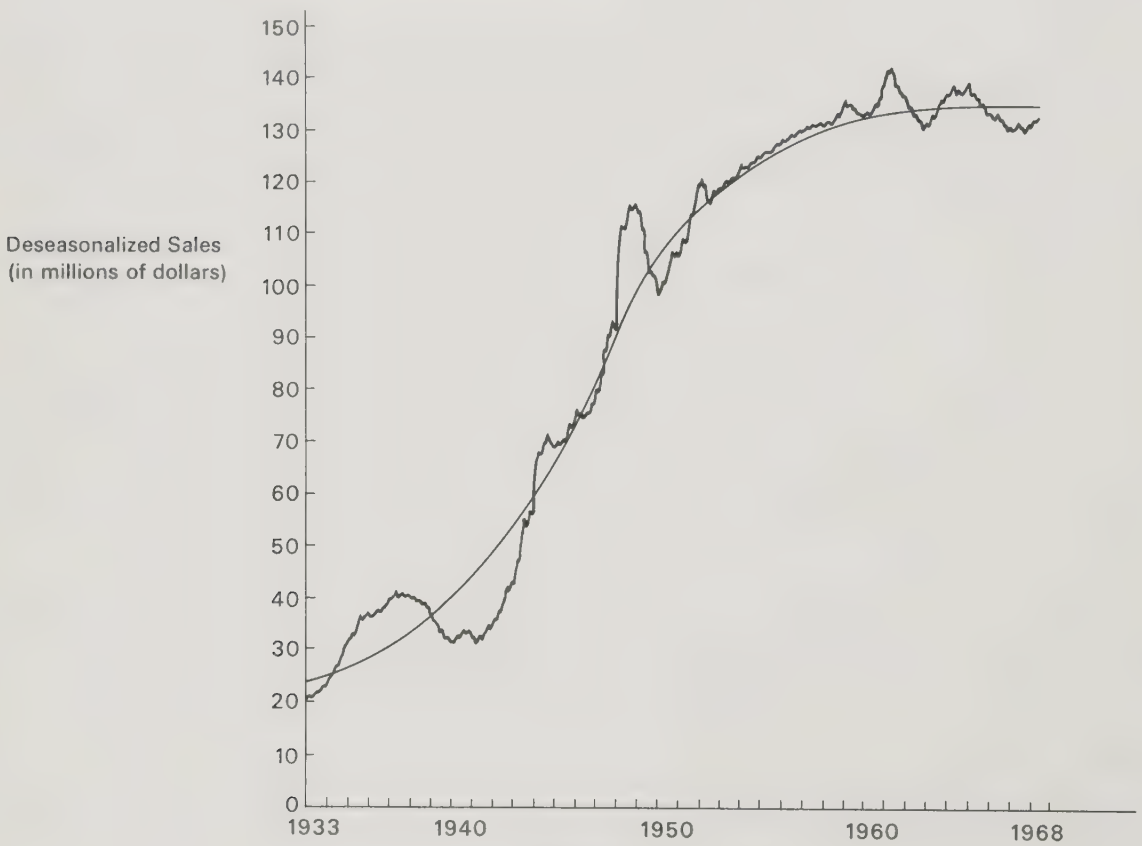



Figure 35.1

Example 2 Refer to Example 1. Figure 35.1 shows the deseasonalized quarterly sales data of the milling company for the period 1933–1968 inclusive.

The fitted trend line is also shown in Figure 35.1. Some of the deseasonalized data and corresponding trend values are presented in Table 35.3. Most of the data have been omitted for conciseness.

**Table 35.3**

| Year and Quarter | Deseasonalized Sales ($ millions) | Trend Value ($ millions) | Per cent of Trend |
|---|---|---|---|
| 1933 1 | 20.7 | 24.3 | 85 |
| 2 | 20.8 | 24.8 | 84 |
| 3 | 21.0 | 25.4 | 83 |
| 4 | 21.4 | 26.0 | 82 |
| 1934 1 | 24.0 | 26.6 | — |
| 2 | 26.7 | 27.3 | — |
| . | . | . | . |
| . | . | . | . |
| . | . | . | . |
| 1967 3 | 128.1 | 134.8 | 95 |
| 4 | 131.4 | 134.9 | 97 |
| 1968 1 | 131.2 | 135.1 | 97 |
| 2 | 131.8 | 135.2 | 97 |
| 3 | 131.9 | 135.3 | — |
| 4 | 132.9 | 135.5 | — |

To illustrate the calculation of a per cent of trend, consider the first quarter of 1933. The ratio of the deseasonalized sales amount to the trend value for that quarter is 20.7/24.3 = .85 or 85 per cent of trend. Per cents of trend for other periods are calculated in a similar manner.

**Problem 2**

Refer to Example 2.

(a) Find the per cents of trend for the first and second quarters of 1934 and for the third and fourth quarters of 1968 in Table 35.3.

(b) Interpret the meaning of the per cent of trend value of 85 for the first quarter of 1933. _____

_____ √

**C. Pattern of Cyclical Fluctuations.** If the cyclical-random component of a time series (as measured by per cents of trend) is plotted, we usually obtain a series of wave-like fluctuations or cycles.

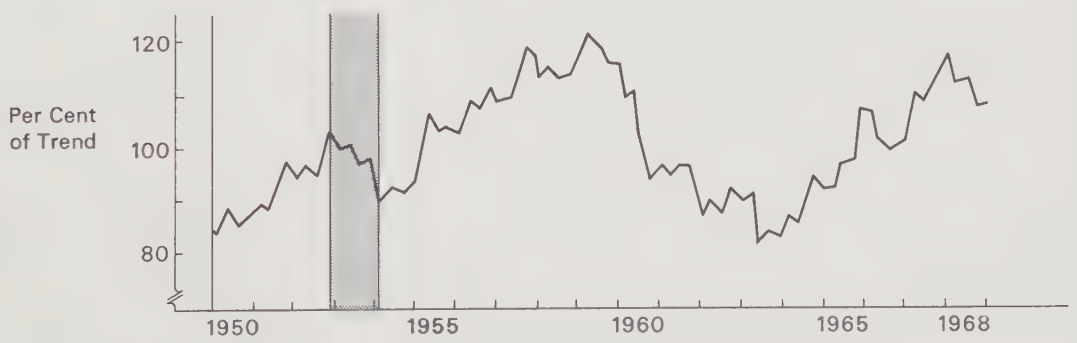



Figure 35.2

Answers

1.

| Year and Quarter | | Deseasonalized Sales |
|---|---|---|
| 1934 | 1 | 26.4/1.10 = 24.0 |
| | 2 | 26.7 |
| 1968 | 3 | 131.9 |
| | 4 | 132.9 |

2. (a)

| Year and Quarter | | Per Cent of Trend |
|---|---|---|
| 1934 | 1 | 90 |
| | 2 | 98 |
| 1968 | 3 | 97 |
| | 4 | 98 |

(b) **A per cent of trend value of 85 for the first quarter of 1933 means that the deseasonalized sales for that period were 85 per cent as great as the trend value for that time period.**

Example 3 Shown in Figure 35.2 on page 307 are quarterly per cents of trend for total production in a certain industry during the period 1950–1968 inclusive.

The cyclical pattern in Figure 35.2 may be divided into successive short-term phases of cyclical expansion and contraction. This division is somewhat arbitrary and hence there may be some disagreement as to the duration of these phases. In Figure 35.2, a period of expansion, ending in the latter part of 1952, is denoted by a clear interval. This is followed by a period of contraction, ending in early 1954, which is denoted by a shaded interval.

**Problem 3**

(a) In Figure 35.2, identify (by shading the intervals of contraction), the periods of cyclical expansion and contraction for the years 1954–1968 inclusive.

(b) Is the cyclical pattern regular enough so that one can tell whether 1972 will be a year of contracting production in the industry? _____

_____

_____ √

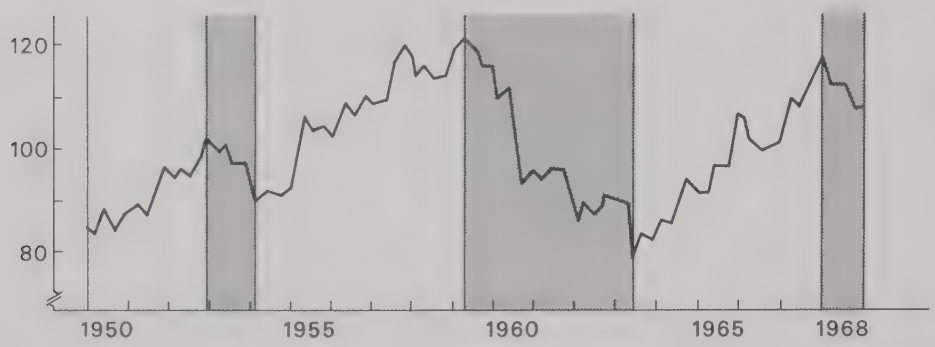
## REVIEW PROBLEMS

1. Refer to Figure 35.2. The per cents of trend in 1963 are less than the per cents of trend in 1958. Does this mean that total production in the industry during 1958 was greater than total production during 1963?

2. If two different trend curves, such as a straight line and a semilogarithmic curve, were fitted to the same deseasonalized time series data, would the per cents of trend for the two curves exhibit the same pattern of cyclical fluctuations?

Answers

3. (a) You should have approximately the following patterns of cyclical expansion and contraction. From the graph alone, it is not certain that a contraction began in 1968. This could be told more clearly after a few years.




(b) Variation in the cyclical pattern and the presence of irregular movements in the cyclical-random component of a time series make it virtually impossible to identify future cyclical behavior by an inspection of the data alone. Thus, a statement about whether or not industry production will contract in 1972 is likely to be little more than a guess, unless it is based on facts and concepts other than the historical movements of the time series.

## ANSWERS TO REVIEW PROBLEMS

1. Per cents of trend describe relative deviations of the actual values from the trend curve, and thus provide no information about the actual values themselves. In Figure 35.2, if the trend curve was sharply upward between 1958 and 1963, it could be that total production during 1963 was greater than total production during 1958.

2. If the two trend curves do not yield identical fits to the data, the trend values in at least some of the time periods will differ and hence per cents of trend will differ for the two trend curves. Different per cents of trend produce somewhat different patterns of cyclical fluctuations. However, if the alternative trends yield reasonable fits, the cyclical patterns will not be grossly dissimilar.

# 36

# Additional Forecasting Techniques

In this section, two additional forecasting techniques are discussed, regression analysis and exponential smoothing.

**A. Regression Analysis.** Regression analysis is a useful technique for studying the statistical relationship between an activity to be forecast (dependent variable) and one or more predictor variables (independent variables). The use of regression analysis for forecasting usually requires that reliable forecasts for independent variables be obtained or that the current values of the independent variables can be used to predict future levels of the dependent variable. Here, only the simple case involving a linear relationship between one independent variable and the activity to be forecast will be considered. (Simple regression analysis is discussed in Section 26. Multiple and curvilinear regression analysis is discussed in Section 27.)

Example 1 A manufacturing firm forecasts sales of several of its products on the basis of disposable personal income. Shown in Figure 36.1 and Table 36.1 are annual sales of one such product in a marketing region, and disposable personal income in the region during 1961–1968 inclusive.

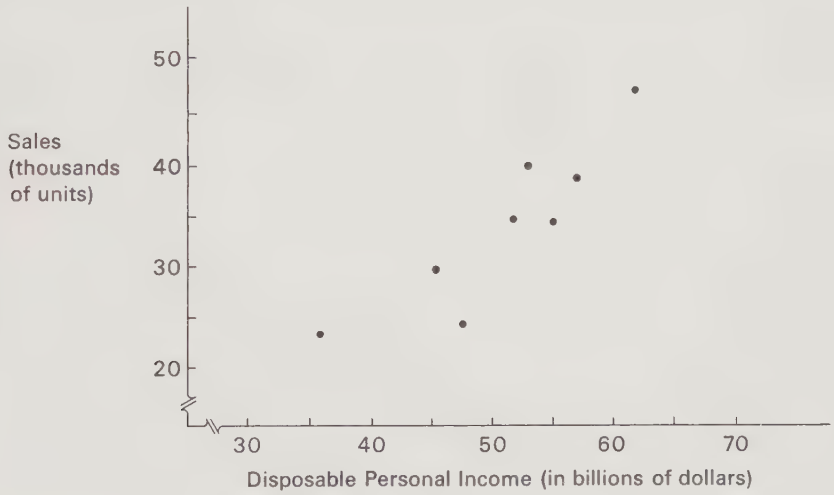



**Figure 36.1**

The scatter diagram (Figure 36.1) indicates that a linear regression function would be a reasonable description of the relationship between the two series. It will be recalled that the linear regression relationship is of the form $\overline{Y}_X = a + bX$, where $a$ and $b$ are obtained as follows:

$$b = \frac{n\Sigma XY - (\Sigma X)(\Sigma Y)}{n\Sigma X^2 - (\Sigma X)(\Sigma X)} = \frac{8(14{,}040.0) - (407.5)(268)}{8(21{,}173.07) - (407.5)(407.5)} = .9344;$$

$$a = \overline{Y} - b\overline{X} = 33.50 - .9344(50.94) = -14.10.$$

Table 36.1

| Year | Disposable Personal Income ($ billions) $X$ | Sales (thousands of units) $Y$ | $XY$ | $X^2$ |
|---|---|---|---|---|
| 1961 | 36.4 | 23 | 837.2 | 1324.96 |
| 1962 | 45.6 | 29 | 1322.4 | 2079.36 |
| 1963 | 47.6 | 24 | 1142.4 | 2265.76 |
| 1964 | 51.7 | 34 | 1757.8 | 2672.89 |
| 1965 | 52.9 | 39 | 2063.1 | 2798.41 |
| 1966 | 55.0 | 34 | 1870.0 | 3025.00 |
| 1967 | 57.0 | 38 | 2166.0 | 3249.00 |
| 1968 | 61.3 | 47 | 2881.1 | 3757.69 |
| Total | 407.5 | 268 | 14,040.0 | 21,173.07 |
| Mean | 50.94 | 33.50 | | |

Hence, the estimated linear regression equation is: $\overline{Y}_X = -14.10 + .9344X$.

**Problem 1**

(a) Plot the estimated regression equation in Example 1 on the scatter diagram.

(b) Is time an explicit variable in the regression equation?_____Explain._____

(c) Early in 1969, the company obtained a reliable forecast of 1969 disposable personal income for the marketing region, which stated that it would be $63.1 billion. Using the regression equation of Example 1, estimate the average sales level when disposable personal income is $X$ = $63.1 billion. _____

(d) What are the main assumptions involved in using the average sales calculated in (c) as a forecast of actual sales in 1969? _____

(e) Does the fact that the 1969 sales level is being predicted from a regression relationship using a level of disposable personal income which is outside the range of past data have any influence on the validity of the sales forecast? _____ √

**B. Exponential Smoothing.** Exponential smoothing is a forecasting technique which is easy to apply and capable of being handled routinely by electronic computers with a minimum of attention and data storage. Here we illustrate the use of exponential smoothing in predicting demand for an item.

The basic forecasting formula of exponential smoothing can be written as follows:

$$\text{new forecast} = \text{current forecast} + w\,(\text{current demand} - \text{current forecast}),$$

where $w$ is a weight that has a value somewhere between 0 and 1. By "new forecast" we mean the forecast made in the current period and applicable to the forthcoming period, by "current forecast" we mean the forecast made in the previous period and applicable to the current period, and by "current demand" we mean demand for the item in the current period.

Example 2 In period one, a demand for 142 units of a certain item was predicted for period two. Actual period two demand was 153 units. Using a weight of $w = .20$, let us forecast the demand for the item in period three using the exponential-smoothing formula. Substituting into the formula, we obtain:

$$\text{new forecast (for period three demand)} = 142 + .2(153 - 142) = 144.2.$$

Thus, the demand forecast for the item in period three is 144 units.

**Problem 2**

Continuing with the illustration in Example 2, suppose actual period three demand was 136. Compute a forecast of demand for period four, using the exponential-smoothing formula with $w = .2$. _____ √

**Problem 3**

A firm experienced the following quarterly demand pattern for an item during 1967 and 1968:

| Year and | 1967 | | | | 1968 | | | | 1969 |
|---|---|---|---|---|---|---|---|---|---|
| Quarter: | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 |
| Demand (in units): | 180 | 168 | 159 | 175 | 190 | 205 | 180 | 182 | ? |
| Forecast (in units): | 175 | ___ | ___ | ___ | ___ | ___ | ___ | ___ | ___ |

(a) Compute the quarterly forecasts that would have been made by means of the basic exponential-smoothing formula, assuming that the forecast of demand for the first quarter of 1967 was 175 units. Use a weight of $w = .10$.√

(b) Shown below are the quarterly forecasts for the same data as in (a) which would have been obtained with a weight of $w = .50$:

| Year and | 1967 | | | | 1968 | | | | 1969 |
|---|---|---|---|---|---|---|---|---|---|
| Quarter: | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 | 1 |
| Demand (in units): | 180 | 168 | 159 | 175 | 190 | 205 | 180 | 182 | ? |
| Forecast (in units): | 175 | 178 | 173 | 166 | 170 | 180 | 193 | 186 | 184 |

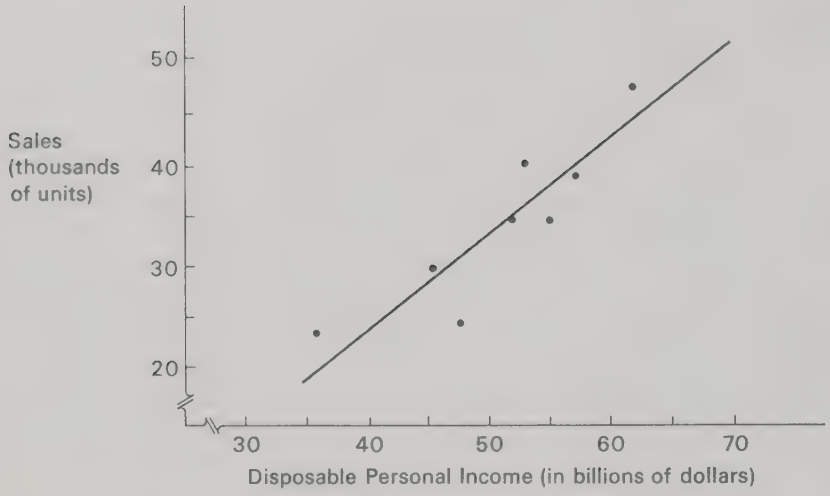
Does an increase in the weight $w$ make the forecasting system more sensitive or less sensitive to changes in demand? _____ Why? _____ √

(c) If demand were chiefly affected by systematic changes in the rate of demand and very little by random movements, would you use a relatively high value of $w$ or a relatively low value? _____

(d) If demand were chiefly affected by random movements and very little by systematic changes in the rate of demand, would you use a relatively high value of $w$ or a relatively low value? _____ √

## Answers

**1. (a)**




**(b)** No, time only enters in that $X$ and $Y$ are associated in time.

**(c)** $\overline{Y}_X = -14.10 + .9344(63.1) = 44.9$ or 44,900 units.

**(d)** (1) The linear regression relationship between disposable personal income and sales during 1961–1968 will be applicable in 1969. (2) The 1969 forecast of disposable personal income is accurate. (3) The deviation between actual sales and the corresponding point on the regression line is relatively small.

**(e)** When predicting beyond the range of past data, extra care must be taken to ascertain that the linear regression relationship continues to be valid. Otherwise, the use of the regression relation can lead to results seriously in error. In this case, the $X$ value does not fall far beyond the range of past experience.

**2. New forecast (for period four demand) = 144.2 + .2(136 − 144.2) = 142.6, or 143 units.**

**3. (a)**

| Year and | 1967 | | | | 1968 | | | |
|---|---|---|---|---|---|---|---|---|
| Quarter: | 1 | 2 | 3 | 4 | 1 | 2 | 3 | 4 |
| Forecast: | 175.00 | 175.50 | 174.75 | 173.18 | 173.36 | 175.00 | 178.00 | 178.20 |

| Year and | 1969 |
|---|---|
| Quarter: | 1 |
| Forecast: | 178.58 |

**(b)** An increase in the weight $w$ makes the forecasting system more sensitive; the larger the value of $w$, the greater the weight given to current demand in computing the new forecast. This can be seen by rewriting the basic forecasting formula: new forecast = $(1 - w)$ (current forecast) + $(w)$ (current demand).

**(c)** A relatively high value of $w$ would be used in order to make the system sensitive to the systematic changes.

**(d)** A relatively low value of $w$ would be used because the system should not be sensitive to random changes.

## REVIEW PROBLEMS

1. The linear regression relationship between the price of corn in a given year and the total production of corn in the previous year is used by an economist to forecast corn prices. This is an example of a *lead-lag relationship*. Price and production data for 1960–1968 inclusive are given in Table 36.2.

**Table 36.2**

| Production Year | Total Production (millions of bushels) $X$ | Price Year | Price ($ per bushel) $Y$ |
|---|---|---|---|
| 1960 | 625 | 1961 | 1.09 |
| 1961 | 490 | 1962 | 1.28 |
| 1962 | 719 | 1963 | 1.05 |
| 1963 | 660 | 1964 | 1.10 |
| 1964 | 585 | 1965 | 1.18 |
| 1965 | 778 | 1966 | 1.02 |
| 1966 | 614 | 1967 | 1.20 |
| 1967 | 588 | 1968 | 1.25 |

Using the method of least squares, the economist has computed the following estimate of the regression equation: $\overline{Y}_X = 1.77 - .000990X$.

(a) If $X$ pertains to year $t$, to what year does $\overline{Y}_X$ pertain? _____

(b) To forecast next year's price with this approach, is it necessary to forecast production? _____

_____

(c) Obtain an estimate of the expected price of corn in 1969 if 730 million bushels of corn are produced in 1968. _____√

2. Sales of a new brand of toothpaste have grown steadily during the five months since its introduction. The sales data (in case lots) are shown below:

| Month: | January | February | March | April | May | June |
|---|---|---|---|---|---|---|
| Sales: | 450 | 495 | 518 | 563 | 584 | ? |
| Forecast: | 410 | ___ | ___ | ___ | ___ | ___ |

The marketing-research director had predicted in December, before the toothpaste was introduced, that January sales would be 410 case lots.

(a) Suppose the basic exponential-smoothing formula is used to obtain monthly forecasts of demand. Use a weight factor of .30 to develop the demand forecasts for February through June inclusive. √

(b) If the marketing-research director's forecast had been 450 rather than 410, would the forecast of demand for June be the same as that obtained in (a)? Explain, referring specifically to the exponential-smoothing forecasting formula. _____

_____

_____

_____

(c) Where demand is growing steadily, as it is in this case, does the basic formula for exponential smoothing provide suitable forecasts? _____

_____

(d) How much past data need be kept in the records or in the computer memory in order to make forecasts with exponential smoothing? _____

_____√

## ANSWERS TO REVIEW PROBLEMS

1. (a) If $X$ pertains to year $t$, $\overline{Y}_X$ pertains to year $t + 1$.
   (b) No, it is not necessary to forecast production since production for this year is related to next year's price.
   (c) $\overline{Y}_X = 1.77 - .000990(730) = \$1.04$ per bushel.

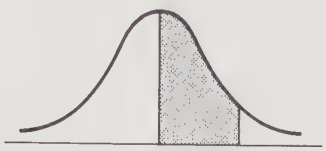
2. (a)

| Month: | January | February | March | April | May | June |
|---|---|---|---|---|---|---|
| Forecast: | 410.00 | 422.00 | 443.90 | 466.13 | 495.19 | 521.83 |

   (b) The forecasts would differ since the forecast applicable to a given period consolidates the initial forecast and demand experience in previous periods. In particular, with a January forecast of 450, the June forecast would be 528.55 rather than 521.83. Note, however, that a change of 40 in the January forecast results in a change of only 6.72 in the June forecast. This fact illustrates the diminishing influence of past demand and forecast values on the current and new forecasts, the farther back these past values are.
   (c) The forecasts in (a) consistently understate the corresponding demands. More refined exponential-smoothing formulas are available to take trend and seasonal movements into account. Taking trend into account would substantially improve the forecasts in the present case.
   (d) The only data that need be kept are the actual demands and forecasts for the current period.

# APPENDIX

## TABLE A-1 Table of Areas for Standard Normal Probability Distribution




| z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .0000 | .0040 | .0080 | .0120 | .0160 | .0199 | .0239 | .0279 | .0319 | .0359 |
| 0.1 | .0398 | .0438 | .0478 | .0517 | .0557 | .0596 | .0636 | .0675 | .0714 | .0753 |
| 0.2 | .0793 | .0832 | .0871 | .0910 | .0948 | .0987 | .1026 | .1064 | .1103 | .1141 |
| 0.3 | .1179 | .1217 | .1255 | .1293 | .1331 | .1368 | .1406 | .1443 | .1480 | .1517 |
| 0.4 | .1554 | .1591 | .1628 | .1664 | .1700 | .1736 | .1772 | .1808 | .1844 | .1879 |
| 0.5 | .1915 | .1950 | .1985 | .2019 | .2054 | .2088 | .2123 | .2157 | .2190 | .2224 |
| 0.6 | .2257 | .2291 | .2324 | .2357 | .2389 | .2422 | .2454 | .2486 | .2518 | .2549 |
| 0.7 | .2580 | .2612 | .2642 | .2673 | .2704 | .2734 | .2764 | .2794 | .2823 | .2852 |
| 0.8 | .2881 | .2910 | .2939 | .2967 | .2995 | .3023 | .3051 | .3078 | .3106 | .3133 |
| 0.9 | .3159 | .3186 | .3212 | .3238 | .3264 | .3289 | .3315 | .3340 | .3365 | .3389 |
| 1.0 | .3413 | .3438 | .3461 | .3485 | .3508 | .3531 | .3554 | .3577 | .3599 | .3621 |
| 1.1 | .3643 | .3665 | .3686 | .3708 | .3729 | .3749 | .3770 | .3790 | .3810 | .3830 |
| 1.2 | .3849 | .3869 | .3888 | .3907 | .3925 | .3944 | .3962 | .3980 | .3997 | .4015 |
| 1.3 | .4032 | .4049 | .4066 | .4082 | .4099 | .4115 | .4131 | .4147 | .4162 | .4177 |
| 1.4 | .4192 | .4207 | .4222 | .4236 | .4251 | .4265 | .4279 | .4292 | .4306 | .4319 |
| 1.5 | .4332 | .4345 | .4357 | .4370 | .4382 | .4394 | .4406 | .4418 | .4429 | .4441 |
| 1.6 | .4452 | .4463 | .4474 | .4484 | .4495 | .4505 | .4515 | .4525 | .4535 | .4545 |
| 1.7 | .4554 | .4564 | .4573 | .4582 | .4591 | .4599 | .4608 | .4616 | .4625 | .4633 |
| 1.8 | .4641 | .4649 | .4656 | .4664 | .4671 | .4678 | .4686 | .4693 | .4699 | .4706 |
| 1.9 | .4713 | .4719 | .4726 | .4732 | .4738 | .4744 | .4750 | .4756 | .4761 | .4767 |
| 2.0 | .4772 | .4778 | .4783 | .4788 | .4793 | .4798 | .4803 | .4808 | .4812 | .4817 |
| 2.1 | .4821 | .4826 | .4830 | .4834 | .4838 | .4842 | .4846 | .4850 | .4854 | .4857 |
| 2.2 | .4861 | .4864 | .4868 | .4871 | .4875 | .4878 | .4881 | .4884 | .4887 | .4890 |
| 2.3 | .4893 | .4896 | .4898 | .4901 | .4904 | .4906 | .4909 | .4911 | .4913 | .4916 |
| 2.4 | .4918 | .4920 | .4922 | .4925 | .4927 | .4929 | .4931 | .4932 | .4934 | .4936 |
| 2.5 | .4938 | .4940 | .4941 | .4943 | .4945 | .4946 | .4948 | .4949 | .4951 | .4952 |
| 2.6 | .4953 | .4955 | .4956 | .4957 | .4959 | .4960 | .4961 | .4962 | .4963 | .4964 |
| 2.7 | .4965 | .4966 | .4967 | .4968 | .4969 | .4970 | .4971 | .4972 | .4973 | .4974 |
| 2.8 | .4974 | .4975 | .4976 | .4977 | .4977 | .4978 | .4979 | .4979 | .4980 | .4981 |
| 2.9 | .4981 | .4982 | .4982 | .4983 | .4984 | .4984 | .4985 | .4985 | .4986 | .4986 |
| 3.0 | .49865 | .4987 | .4987 | .4988 | .4988 | .4989 | .4989 | .4989 | .4990 | .4990 |
| 4.0 | .4999683 | | | | | | | | | |

*Illustration:* For $z = 1.93$, shaded area is .4732 out of total area of 1.

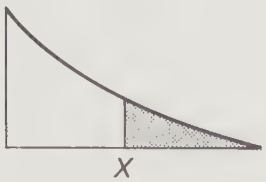
## TABLE A-2 Poisson Probabilities

| | $\mu$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| 0 | .9048 | .8187 | .7408 | .6703 | .6065 | .5488 | .4966 | .4493 | .4066 | .3679 |
| 1 | .0905 | .1637 | .2222 | .2681 | .3033 | .3293 | .3476 | .3595 | .3659 | .3679 |
| 2 | .0045 | .0164 | .0333 | .0536 | .0758 | .0988 | .1217 | .1438 | .1647 | .1839 |
| 3 | .0002 | .0011 | .0033 | .0072 | .0126 | .0198 | .0284 | .0383 | .0494 | .0613 |
| 4 | | .0001 | .0002 | .0007 | .0016 | .0030 | .0050 | .0077 | .0111 | .0153 |
| 5 | | | | .0001 | .0002 | .0004 | .0007 | .0012 | .0020 | .0031 |
| 6 | | | | | | | .0001 | .0002 | .0003 | .0005 |
| 7 | | | | | | | | | | .0001 |

| | $\mu$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $X$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 | 5.0 | 6.0 | 7.0 |
| 0 | .2231 | .1353 | .0821 | .0498 | .0302 | .0183 | .0111 | .0067 | .0025 | .0009 |
| 1 | .3347 | .2707 | .2052 | .1494 | .1057 | .0733 | .0500 | .0337 | .0149 | .0064 |
| 2 | .2510 | .2707 | .2565 | .2240 | .1850 | .1465 | .1125 | .0842 | .0446 | .0223 |
| 3 | .1255 | .1804 | .2138 | .2240 | .2158 | .1954 | .1687 | .1404 | .0892 | .0521 |
| 4 | .0471 | .0902 | .1336 | .1680 | .1888 | .1954 | .1898 | .1755 | .1339 | .0912 |
| 5 | .0141 | .0361 | .0668 | .1008 | .1322 | .1563 | .1708 | .1755 | .1606 | .1277 |
| 6 | .0035 | .0120 | .0278 | .0504 | .0771 | .1042 | .1281 | .1462 | .1606 | .1490 |
| 7 | .0008 | .0034 | .0099 | .0216 | .0385 | .0595 | .0824 | .1044 | .1377 | .1490 |
| 8 | .0001 | .0009 | .0031 | .0081 | .0169 | .0298 | .0463 | .0653 | .1033 | .1304 |
| 9 | | .0002 | .0009 | .0027 | .0066 | .0132 | .0232 | .0363 | .0688 | .1014 |
| 10 | | | .0002 | .0008 | .0023 | .0053 | .0104 | .0181 | .0413 | .0710 |
| 11 | | | | .0002 | .0007 | .0019 | .0043 | .0082 | .0225 | .0452 |
| 12 | | | | .0001 | .0002 | .0006 | .0016 | .0034 | .0113 | .0264 |
| 13 | | | | | .0001 | .0002 | .0006 | .0013 | .0052 | .0142 |
| 14 | | | | | | .0001 | .0002 | .0005 | .0022 | .0071 |
| 15 | | | | | | | .0001 | .0002 | .0009 | .0033 |
| 16 | | | | | | | | | .0003 | .0014 |
| 17 | | | | | | | | | .0001 | .0006 |
| 18 | | | | | | | | | | .0002 |
| 19 | | | | | | | | | | .0001 |

*Example:* If $\mu = 1$, $P(X = 2) = .1839$.

## TABLE A-3 Probabilities in Right Tail of Exponential Probability Distribution




| $\frac{X}{\mu}$ | Prob. | $\frac{X}{\mu}$ | Prob. | $\frac{X}{\mu}$ | Prob. | $\frac{X}{\mu}$ | Prob. |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.000 | 2.5 | 0.082 | 5.0 | 0.0067 | 7.5 | 0.00055 |
| 0.1 | 0.905 | 2.6 | 0.074 | 5.1 | 0.0061 | 7.6 | 0.00050 |
| 0.2 | 0.819 | 2.7 | 0.067 | 5.2 | 0.0055 | 7.7 | 0.00045 |
| 0.3 | 0.741 | 2.8 | 0.061 | 5.3 | 0.0050 | 7.8 | 0.00041 |
| 0.4 | 0.670 | 2.9 | 0.055 | 5.4 | 0.0045 | 7.9 | 0.00037 |
| 0.5 | 0.607 | 3.0 | 0.050 | 5.5 | 0.0041 | 8.0 | 0.00034 |
| 0.6 | 0.549 | 3.1 | 0.045 | 5.6 | 0.0037 | 8.1 | 0.00030 |
| 0.7 | 0.497 | 3.2 | 0.041 | 5.7 | 0.0033 | 8.2 | 0.00028 |
| 0.8 | 0.449 | 3.3 | 0.037 | 5.8 | 0.0030 | 8.3 | 0.00025 |
| 0.9 | 0.407 | 3.4 | 0.033 | 5.9 | 0.0027 | 8.4 | 0.00022 |
| 1.0 | 0.368 | 3.5 | 0.030 | 6.0 | 0.0025 | 8.5 | 0.00020 |
| 1.1 | 0.333 | 3.6 | 0.027 | 6.1 | 0.0022 | 8.6 | 0.00018 |
| 1.2 | 0.301 | 3.7 | 0.025 | 6.2 | 0.0020 | 8.7 | 0.00017 |
| 1.3 | 0.273 | 3.8 | 0.022 | 6.3 | 0.0018 | 8.8 | 0.00015 |
| 1.4 | 0.247 | 3.9 | 0.020 | 6.4 | 0.0017 | 8.9 | 0.00014 |
| 1.5 | 0.223 | 4.0 | 0.018 | 6.5 | 0.0015 | 9.0 | 0.00012 |
| 1.6 | 0.202 | 4.1 | 0.017 | 6.6 | 0.0014 | 9.1 | 0.00011 |
| 1.7 | 0.183 | 4.2 | 0.015 | 6.7 | 0.0012 | 9.2 | 0.00010 |
| 1.8 | 0.165 | 4.3 | 0.014 | 6.8 | 0.0011 | 9.3 | 0.00009 |
| 1.9 | 0.150 | 4.4 | 0.012 | 6.9 | 0.0010 | 9.4 | 0.00008 |
| 2.0 | 0.135 | 4.5 | 0.011 | 7.0 | 0.0009 | 9.5 | 0.00008 |
| 2.1 | 0.122 | 4.6 | 0.010 | 7.1 | 0.0008 | 9.6 | 0.00007 |
| 2.2 | 0.111 | 4.7 | 0.009 | 7.2 | 0.0007 | 9.7 | 0.00006 |
| 2.3 | 0.100 | 4.8 | 0.008 | 7.3 | 0.0007 | 9.8 | 0.00006 |
| 2.4 | 0.091 | 4.9 | 0.007 | 7.4 | 0.0006 | 9.9 | 0.00005 |

*Illustration:* If $\mu = 600$, the probability of exceeding $X = 900$ is .223.

## TABLE A-4 Binomial Probabilities

| $n$ | $\bar{p}$ | $p$ .05 | .10 | .15 | .20 | .25 | .30 | .35 | .40 | .45 | .50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0 | .9025 | .8100 | .7225 | .6400 | .5625 | .4900 | .4225 | .3600 | .3025 | .2500 |
| | $\frac{1}{2}$ | .0950 | .1800 | .2550 | .3200 | .3750 | .4200 | .4550 | .4800 | .4950 | .5000 |
| | 1 | .0025 | .0100 | .0225 | .0400 | .0625 | .0900 | .1225 | .1600 | .2025 | .2500 |
| 3 | 0 | .8574 | .7290 | .6141 | .5120 | .4219 | .3430 | .2746 | .2160 | .1664 | .1250 |
| | $\frac{1}{3}$ | .1354 | .2430 | .3251 | .3840 | .4219 | .4410 | .4436 | .4320 | .4084 | .3750 |
| | $\frac{2}{3}$ | .0071 | .0270 | .0574 | .0960 | .1406 | .1890 | .2389 | .2880 | .3341 | .3750 |
| | 1 | .0001 | .0010 | .0034 | .0080 | .0156 | .0270 | .0429 | .0640 | .0911 | .1250 |
| 4 | 0 | .8145 | .6561 | .5220 | .4096 | .3164 | .2401 | .1785 | .1296 | .0915 | .0625 |
| | $\frac{1}{4}$ | .1715 | .2916 | .3685 | .4096 | .4219 | .4116 | .3845 | .3456 | .2995 | .2500 |
| | $\frac{2}{4}$ | .0135 | .0486 | .0975 | .1536 | .2109 | .2646 | .3105 | .3456 | .3675 | .3750 |
| | $\frac{3}{4}$ | .0005 | .0036 | .0115 | .0256 | .0469 | .0756 | .1115 | .1536 | .2005 | .2500 |
| | 1 | | .0001 | .0005 | .0016 | .0039 | .0081 | .0150 | .0256 | .0410 | .0625 |
| 5 | 0 | .7738 | .5905 | .4437 | .3277 | .2373 | .1681 | .1160 | .0778 | .0503 | .0312 |
| | $\frac{1}{5}$ | .2036 | .3280 | .3915 | .4096 | .3955 | .3602 | .3124 | .2592 | .2059 | .1562 |
| | $\frac{2}{5}$ | .0214 | .0729 | .1382 | .2048 | .2637 | .3087 | .3364 | .3456 | .3369 | .3125 |
| | $\frac{3}{5}$ | .0011 | .0081 | .0244 | .0512 | .0879 | .1323 | .1811 | .2304 | .2757 | .3125 |
| | $\frac{4}{5}$ | | .0004 | .0022 | .0064 | .0146 | .0284 | .0488 | .0768 | .1128 | .1562 |
| | 1 | | | .0001 | .0003 | .0010 | .0024 | .0053 | .0102 | .0185 | .0312 |
| 6 | 0 | .7351 | .5314 | .3771 | .2621 | .1780 | .1176 | .0754 | .0467 | .0277 | .0156 |
| | $\frac{1}{6}$ | .2321 | .3543 | .3993 | .3932 | .3560 | .3025 | .2437 | .1866 | .1359 | .0938 |
| | $\frac{2}{6}$ | .0305 | .0984 | .1762 | .2458 | .2966 | .3241 | .3280 | .3110 | .2780 | .2344 |
| | $\frac{3}{6}$ | .0021 | .0146 | .0415 | .0819 | .1318 | .1852 | .2355 | .2765 | .3032 | .3125 |
| | $\frac{4}{6}$ | .0001 | .0012 | .0055 | .0154 | .0330 | .0595 | .0951 | .1382 | .1861 | .2344 |
| | $\frac{5}{6}$ | | .0001 | .0004 | .0015 | .0044 | .0102 | .0205 | .0369 | .0609 | .0938 |
| | 1 | | | | .0001 | .0002 | .0007 | .0018 | .0041 | .0083 | .0156 |
| 7 | 0 | .6983 | .4783 | .3206 | .2097 | .1335 | .0824 | .0490 | .0280 | .0152 | .0078 |
| | $\frac{1}{7}$ | .2573 | .3720 | .3960 | .3670 | .3115 | .2471 | .1848 | .1306 | .0872 | .0547 |
| | $\frac{2}{7}$ | .0406 | .1240 | .2097 | .2753 | .3115 | .3177 | .2985 | .2613 | .2140 | .1641 |
| | $\frac{3}{7}$ | .0036 | .0230 | .0617 | .1147 | .1730 | .2269 | .2679 | .2903 | .2918 | .2734 |
| | $\frac{4}{7}$ | .0002 | .0026 | .0109 | .0287 | .0577 | .0972 | .1442 | .1935 | .2388 | .2734 |
| | $\frac{5}{7}$ | | .0002 | .0012 | .0043 | .0115 | .0250 | .0466 | .0774 | .1172 | .1641 |
| | $\frac{6}{7}$ | | | .0001 | .0004 | .0013 | .0036 | .0084 | .0172 | .0320 | .0547 |
| | 1 | | | | | .0001 | .0002 | .0006 | .0016 | .0037 | .0078 |
| 8 | 0 | .6634 | .4305 | .2725 | .1678 | .1001 | .0576 | .0319 | .0168 | .0084 | .0039 |
| | $\frac{1}{8}$ | .2793 | .3826 | .3847 | .3355 | .2670 | .1977 | .1373 | .0896 | .0548 | .0312 |
| | $\frac{2}{8}$ | .0515 | .1488 | .2376 | .2936 | .3115 | .2965 | .2587 | .2090 | .1569 | .1094 |
| | $\frac{3}{8}$ | .0054 | .0331 | .0839 | .1468 | .2076 | .2541 | .2786 | .2787 | .2568 | .2188 |
| | $\frac{4}{8}$ | .0004 | .0046 | .0185 | .0459 | .0865 | .1361 | .1875 | .2322 | .2627 | .2734 |
| | $\frac{5}{8}$ | | .0004 | .0026 | .0092 | .0231 | .0467 | .0808 | .1239 | .1719 | .2188 |
| | $\frac{6}{8}$ | | | .0002 | .0011 | .0038 | .0100 | .0217 | .0413 | .0703 | .1094 |
| | $\frac{7}{8}$ | | | | .0001 | .0004 | .0012 | .0033 | .0079 | .0164 | .0312 |
| | 1 | | | | | | .0001 | .0002 | .0007 | .0017 | .0039 |

*Example:* If $p = .25$, $n = 5$, $P(\bar{p} = 3/5) = .0879$.

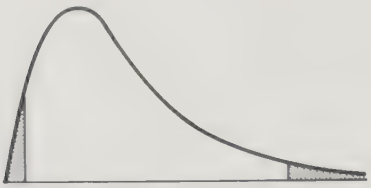
## TABLE A-5 Table of $t$ Distribution

| Degrees of Freedom | Total Area in Both Tails | | | |
|---|---|---|---|---|
| | .10 | .05 | .02 | .01 |
| 1 | 6.314 | 12.706 | 31.821 | 63.657 |
| 2 | 2.920 | 4.303 | 6.965 | 9.925 |
| 3 | 2.353 | 3.182 | 4.541 | 5.841 |
| 4 | 2.132 | 2.776 | 3.747 | 4.604 |
| 5 | 2.015 | 2.571 | 3.365 | 4.032 |
| 6 | 1.943 | 2.447 | 3.143 | 3.707 |
| 7 | 1.895 | 2.365 | 2.998 | 3.499 |
| 8 | 1.860 | 2.306 | 2.896 | 3.355 |
| 9 | 1.833 | 2.262 | 2.821 | 3.250 |
| 10 | 1.812 | 2.228 | 2.764 | 3.169 |
| 11 | 1.796 | 2.201 | 2.718 | 3.106 |
| 12 | 1.782 | 2.179 | 2.681 | 3.055 |
| 13 | 1.771 | 2.160 | 2.650 | 3.012 |
| 14 | 1.761 | 2.145 | 2.624 | 2.977 |
| 15 | 1.753 | 2.131 | 2.602 | 2.947 |
| 16 | 1.746 | 2.120 | 2.583 | 2.921 |
| 17 | 1.740 | 2.110 | 2.567 | 2.898 |
| 18 | 1.734 | 2.101 | 2.552 | 2.878 |
| 19 | 1.729 | 2.093 | 2.539 | 2.861 |
| 20 | 1.725 | 2.086 | 2.528 | 2.845 |
| 21 | 1.721 | 2.080 | 2.518 | 2.831 |
| 22 | 1.717 | 2.074 | 2.508 | 2.819 |
| 23 | 1.714 | 2.069 | 2.500 | 2.807 |
| 24 | 1.711 | 2.064 | 2.492 | 2.797 |
| 25 | 1.708 | 2.060 | 2.485 | 2.787 |
| 26 | 1.706 | 2.056 | 2.479 | 2.779 |
| 27 | 1.703 | 2.052 | 2.473 | 2.771 |
| 28 | 1.701 | 2.048 | 2.467 | 2.763 |
| 29 | 1.699 | 2.045 | 2.462 | 2.756 |
| 30 | 1.697 | 2.042 | 2.457 | 2.750 |
| 40 | 1.684 | 2.021 | 2.423 | 2.704 |
| Normal Distribution | 1.645 | 1.960 | 2.326 | 2.576 |

*Illustration:* The $t$ value for 9 degrees of freedom corresponding to area of .05 in both tails is 2.262.

*Source:* Taken from Table III of Fisher and Yates, *Statistical Tables for Biological, Agricultural, and Medical Research,* 6th ed., 1963, published by Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publishers.

## TABLE A-6 Table of $\chi^2$ Distribution

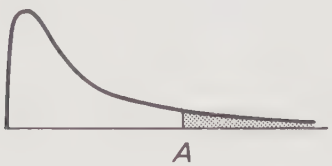


| Degrees of Freedom $\delta$ | Area in Left Tail | | | | Area in Right Tail | | | |
|---|---|---|---|---|---|---|---|---|
| | .01 | .02 | .05 | .10 | .10 | .05 | .02 | .01 |
| 1 | .000 | .001 | .004 | .016 | 2.706 | 3.841 | 5.412 | 6.635 |
| 2 | .020 | .040 | .103 | .211 | 4.605 | 5.991 | 7.824 | 9.210 |
| 3 | .115 | .185 | .352 | .584 | 6.251 | 7.815 | 9.837 | 11.345 |
| 4 | .297 | .429 | .711 | 1.064 | 7.779 | 9.488 | 11.668 | 13.277 |
| 5 | .554 | .752 | 1.145 | 1.610 | 9.236 | 11.070 | 13.388 | 15.086 |
| 6 | .872 | 1.134 | 1.635 | 2.204 | 10.645 | 12.592 | 15.033 | 16.812 |
| 7 | 1.239 | 1.564 | 2.167 | 2.833 | 12.017 | 14.067 | 16.622 | 18.475 |
| 8 | 1.646 | 2.032 | 2.733 | 3.490 | 13.362 | 15.507 | 18.168 | 20.090 |
| 9 | 2.088 | 2.532 | 3.325 | 4.168 | 14.684 | 16.919 | 19.679 | 21.666 |
| 10 | 2.558 | 3.059 | 3.940 | 4.865 | 15.987 | 18.307 | 21.161 | 23.209 |
| 11 | 3.053 | 3.609 | 4.575 | 5.578 | 17.275 | 19.675 | 22.618 | 24.725 |
| 12 | 3.571 | 4.178 | 5.226 | 6.304 | 18.549 | 21.026 | 24.054 | 26.217 |
| 13 | 4.107 | 4.765 | 5.892 | 7.042 | 19.812 | 22.362 | 25.472 | 27.688 |
| 14 | 4.660 | 5.368 | 6.571 | 7.790 | 21.064 | 23.685 | 26.873 | 29.141 |
| 15 | 5.229 | 5.985 | 7.261 | 8.547 | 22.307 | 24.996 | 28.259 | 30.578 |
| 16 | 5.812 | 6.614 | 7.962 | 9.312 | 23.542 | 26.296 | 29.633 | 32.000 |
| 17 | 6.408 | 7.255 | 8.672 | 10.085 | 24.769 | 27.587 | 30.995 | 33.409 |
| 18 | 7.015 | 7.906 | 9.390 | 10.865 | 25.989 | 28.869 | 32.346 | 34.805 |
| 19 | 7.633 | 8.567 | 10.117 | 11.651 | 27.204 | 30.144 | 33.687 | 36.191 |
| 20 | 8.260 | 9.237 | 10.851 | 12.443 | 28.412 | 31.410 | 35.020 | 37.566 |
| 21 | 8.897 | 9.915 | 11.591 | 13.240 | 29.615 | 32.671 | 36.343 | 38.932 |
| 22 | 9.542 | 10.600 | 12.338 | 14.041 | 30.813 | 33.924 | 37.659 | 40.289 |
| 23 | 10.196 | 11.293 | 13.091 | 14.848 | 32.007 | 35.172 | 38.968 | 41.638 |
| 24 | 10.856 | 11.992 | 13.848 | 15.659 | 33.196 | 36.415 | 40.270 | 42.980 |
| 25 | 11.524 | 12.697 | 14.611 | 16.473 | 34.382 | 37.652 | 41.566 | 44.314 |
| 26 | 12.198 | 13.409 | 15.379 | 17.292 | 35.563 | 38.885 | 42.856 | 45.642 |
| 27 | 12.879 | 14.125 | 16.151 | 18.114 | 36.741 | 40.113 | 44.140 | 46.963 |
| 28 | 13.565 | 14.847 | 16.928 | 18.939 | 37.916 | 41.337 | 45.419 | 48.278 |
| 29 | 14.256 | 15.574 | 17.708 | 19.768 | 39.087 | 42.557 | 46.693 | 49.588 |
| 30 | 14.953 | 16.306 | 18.493 | 20.599 | 40.256 | 43.773 | 47.962 | 50.892 |

*Illustration:* The values on a $\chi^2$ scale for $\delta = 4$ associated with equal tail areas of .01 each are .297 and 13.277.

*Source:* From Table IV of Fisher and Yates: *Statistical Tables for Biological, Agricultural and Medical Research,* 6th ed., 1963, published by Oliver & Boyd Ltd., Edinburgh, and by permission of the authors and publishers.

## TABLE A-7 Table of $F$ Distribution




$F$ scale value corresponding to area .05 in right tail in light-face type
**$F$ scale value corresponding to area .01 in right tail in bold-face type**

| $\delta_1$ | $\delta_2$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 1 | 161 | 200 | 216 | 225 | 230 | 234 | 237 | 239 | 241 | 242 |
| | **4,052** | **4,999** | **5,403** | **5,625** | **5,764** | **5,859** | **5,928** | **5,981** | **6,022** | **6,056** |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.36 | 19.37 | 19.38 | 19.39 |
| | **98.49** | **99.00** | **99.17** | **99.25** | **99.30** | **99.33** | **99.36** | **99.37** | **99.39** | **99.40** |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.88 | 8.84 | 8.81 | 8.78 |
| | **34.12** | **30.82** | **29.46** | **28.71** | **28.24** | **27.91** | **27.67** | **27.49** | **27.34** | **27.23** |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 | 5.96 |
| | **21.20** | **18.00** | **16.69** | **15.98** | **15.52** | **15.21** | **14.98** | **14.80** | **14.66** | **14.54** |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.78 | 4.74 |
| | **16.26** | **13.27** | **12.06** | **11.39** | **10.97** | **10.67** | **10.45** | **10.29** | **10.15** | **10.05** |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 | 4.06 |
| | **13.74** | **10.92** | **9.78** | **9.15** | **8.75** | **8.47** | **8.26** | **8.10** | **7.98** | **7.87** |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 | 3.63 |
| | **12.25** | **9.55** | **8.45** | **7.85** | **7.46** | **7.19** | **7.00** | **6.84** | **6.71** | **6.62** |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 | 3.34 |
| | **11.26** | **8.65** | **7.59** | **7.01** | **6.63** | **6.37** | **6.19** | **6.03** | **5.91** | **5.82** |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 | 3.13 |
| | **10.56** | **8.02** | **6.99** | **6.42** | **6.06** | **5.80** | **5.62** | **5.47** | **5.35** | **5.26** |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 | 2.97 |
| | **10.04** | **7.56** | **6.55** | **5.99** | **5.64** | **5.39** | **5.21** | **5.06** | **4.95** | **4.85** |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 | 2.86 |
| | **9.65** | **7.20** | **6.22** | **5.67** | **5.32** | **5.07** | **4.88** | **4.74** | **4.63** | **4.54** |

(*Continued*)

**TABLE A-7 Table of $F$ Distribution (*Continued*)**

| $\delta_1$ | $\delta_2$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 12 | 4.75 | 3.88 | 3.49 | 3.26 | 3.11 | 3.00 | 2.92 | 2.85 | 2.80 | 2.76 |
| | **9.33** | **6.93** | **5.95** | **5.41** | **5.06** | **4.82** | **4.65** | **4.50** | **4.39** | **4.30** |
| 13 | 4.67 | 3.80 | 3.41 | 3.18 | 3.02 | 2.92 | 2.84 | 2.77 | 2.72 | 2.67 |
| | **9.07** | **6.70** | **5.74** | **5.20** | **4.86** | **4.62** | **4.44** | **4.30** | **4.19** | **4.10** |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.77 | 2.70 | 2.65 | 2.60 |
| | **8.86** | **6.51** | **5.56** | **5.03** | **4.69** | **4.46** | **4.28** | **4.14** | **4.03** | **3.94** |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.70 | 2.64 | 2.59 | 2.55 |
| | **8.68** | **6.36** | **5.42** | **4.89** | **4.56** | **4.32** | **4.14** | **4.00** | **3.89** | **3.80** |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 | 2.49 |
| | **8.53** | **6.23** | **5.29** | **4.77** | **4.44** | **4.20** | **4.03** | **3.89** | **3.78** | **3.69** |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.62 | 2.55 | 2.50 | 2.45 |
| | **8.40** | **6.11** | **5.18** | **4.67** | **4.34** | **4.10** | **3.93** | **3.79** | **3.68** | **3.59** |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 | 2.41 |
| | **8.28** | **6.01** | **5.09** | **4.58** | **4.25** | **4.01** | **3.85** | **3.71** | **3.60** | **3.51** |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.55 | 2.48 | 2.43 | 2.38 |
| | **8.18** | **5.93** | **5.01** | **4.50** | **4.17** | **3.94** | **3.77** | **3.63** | **3.52** | **3.43** |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.52 | 2.45 | 2.40 | 2.35 |
| | **8.10** | **5.85** | **4.94** | **4.43** | **4.10** | **3.87** | **3.71** | **3.56** | **3.45** | **3.37** |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 | 2.32 |
| | **8.02** | **5.78** | **4.87** | **4.37** | **4.04** | **3.81** | **3.65** | **3.51** | **3.40** | **3.31** |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.47 | 2.40 | 2.35 | 2.30 |
| | **7.94** | **5.72** | **4.82** | **4.31** | **3.99** | **3.76** | **3.59** | **3.45** | **3.35** | **3.26** |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.45 | 2.38 | 2.32 | 2.28 |
| | **7.88** | **5.66** | **4.76** | **4.26** | **3.94** | **3.71** | **3.54** | **3.41** | **3.30** | **3.21** |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.43 | 2.36 | 2.30 | 2.26 |
| | **7.82** | **5.61** | **4.72** | **4.22** | **3.90** | **3.67** | **3.50** | **3.36** | **3.25** | **3.17** |
| 25 | 4.24 | 3.38 | 2.99 | 2.76 | 2.60 | 2.49 | 2.41 | 2.34 | 2.28 | 2.24 |
| | **7.77** | **5.57** | **4.68** | **4.18** | **3.86** | **3.63** | **3.46** | **3.32** | **3.21** | **3.13** |

*Illustration:* The $F$ scale value for $\delta_2 = 3$, $\delta_1 = 10$ corresponding to area .01 in right tail is 6.55.

*Source:* Abridged by permission from *Statistical Methods,* 6th ed., by George W. Snedecor and William C. Cochran. © 1967 by the Iowa State University Press, Ames, Iowa.

**TABLE A-8 Table of $w$ Transformation of Correlation Coefficient $r$**

| $r$ | $w$ | $r$ | $w$ | $r$ | $w$ | $r$ | $w$ |
|---|---|---|---|---|---|---|---|
| .00 | .0000 | .25 | .2554 | .50 | .5493 | .75 | .973 |
| .01 | .0100 | .26 | .2661 | .51 | .5627 | .76 | .996 |
| .02 | .0200 | .27 | .2769 | .52 | .5763 | .77 | 1.020 |
| .03 | .0300 | .28 | .2877 | .53 | .5901 | .78 | 1.045 |
| .04 | .0400 | .29 | .2986 | .54 | .6042 | .79 | 1.071 |
| .05 | .0500 | .30 | .3095 | .55 | .6184 | .80 | 1.099 |
| .06 | .0601 | .31 | .3205 | .56 | .6328 | .81 | 1.127 |
| .07 | .0701 | .32 | .3316 | .57 | .6475 | .82 | 1.157 |
| .08 | .0802 | .33 | .3428 | .58 | .6625 | .83 | 1.188 |
| .09 | .0902 | .34 | .3541 | .59 | .6777 | .84 | 1.221 |
| .10 | .1003 | .35 | .3654 | .60 | .6931 | .85 | 1.256 |
| .11 | .1104 | .36 | .3769 | .61 | .7089 | .86 | 1.293 |
| .12 | .1206 | .37 | .3884 | .62 | .7250 | .87 | 1.333 |
| .13 | .1307 | .38 | .4001 | .63 | .7414 | .88 | 1.376 |
| .14 | .1409 | .39 | .4118 | .64 | .7582 | .89 | 1.422 |
| .15 | .1511 | .40 | .4236 | .65 | .7753 | .90 | 1.472 |
| .16 | .1614 | .41 | .4356 | .66 | .7928 | .91 | 1.528 |
| .17 | .1717 | .42 | .4477 | .67 | .8107 | .92 | 1.589 |
| .18 | .1820 | .43 | .4599 | .68 | .8291 | .93 | 1.658 |
| .19 | .1923 | .44 | .4722 | .69 | .8480 | .94 | 1.738 |
| .20 | .2027 | .45 | .4847 | .70 | .8673 | .95 | 1.832 |
| .21 | .2132 | .46 | .4973 | .71 | .8872 | .96 | 1.946 |
| .22 | .2237 | .47 | .5101 | .72 | .9076 | .97 | 2.092 |
| .23 | .2342 | .48 | .5230 | .73 | .9287 | .98 | 2.298 |
| .24 | .2448 | .49 | .5361 | .74 | .9505 | .99 | 2.647 |

*Illustration:* The value of $w$ for $r = .76$ is .996.

*Source:* Abridged from Table 14 of Pearson and Hartley, *Biometrika Tables for Statisticians,* Volume 1, 1966, published by the Cambridge University Press, on behalf of The Biometrika Society, by permission of the authors and publishers.

## TABLE A-9 Table of Unit Normal Loss

| $D$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .3989 | .3940 | .3890 | .3841 | .3793 | .3744 | .3697 | .3649 | .3602 | .3556 |
| .1 | .3509 | .3464 | .3418 | .3373 | .3328 | .3284 | .3240 | .3197 | .3154 | .3111 |
| .2 | .3069 | .3027 | .2986 | .2944 | .2904 | .2863 | .2824 | .2784 | .2745 | .2706 |
| .3 | .2668 | .2630 | .2592 | .2555 | .2518 | .2481 | .2445 | .2409 | .2374 | .2339 |
| .4 | .2304 | .2270 | .2236 | .2203 | .2169 | .2137 | .2104 | .2072 | .2040 | .2009 |
| .5 | .1978 | .1947 | .1917 | .1887 | .1857 | .1828 | .1799 | .1771 | .1742 | .1714 |
| .6 | .1687 | .1659 | .1633 | .1606 | .1580 | .1554 | .1528 | .1503 | .1478 | .1453 |
| .7 | .1429 | .1405 | .1381 | .1358 | .1334 | .1312 | .1289 | .1267 | .1245 | .1223 |
| .8 | .1202 | .1181 | .1160 | .1140 | .1120 | .1100 | .1080 | .1061 | .1042 | .1023 |
| .9 | .1004 | .09860 | .09680 | .09503 | .09328 | .09156 | .08986 | .08819 | .08654 | .08491 |
| 1.0 | .08332 | .08174 | .08019 | .07866 | .07716 | .07568 | .07422 | .07279 | .07138 | .06999 |
| 1.1 | .06862 | .06727 | .06595 | .06465 | .06336 | .06210 | .06086 | .05964 | .05844 | .05726 |
| 1.2 | .05610 | .05496 | .05384 | .05274 | .05165 | .05059 | .04954 | .04851 | .04750 | .04650 |
| 1.3 | .04553 | .04457 | .04363 | .04270 | .04179 | .04090 | .04002 | .03916 | .03831 | .03748 |
| 1.4 | .03667 | .03587 | .03508 | .03431 | .03356 | .03281 | .03208 | .03137 | .03067 | .02998 |
| 1.5 | .02931 | .02865 | .02800 | .02736 | .02674 | .02612 | .02552 | .02494 | .02436 | .02380 |
| 1.6 | .02324 | .02270 | .02217 | .02165 | .02114 | .02064 | .02015 | .01967 | .01920 | .01874 |
| 1.7 | .01829 | .01785 | .01742 | .01699 | .01658 | .01617 | .01578 | .01539 | .01501 | .01464 |
| 1.8 | .01428 | .01392 | .01357 | .01323 | .01290 | .01257 | .01226 | .01195 | .01164 | .01134 |
| 1.9 | .01105 | .01077 | .01049 | .01022 | $.0^{2}9957$ | $.0^{2}9698$ | $.0^{2}9445$ | $.0^{2}9198$ | $.0^{2}8957$ | $.0^{2}8721$ |
| 2.0 | $.0^{2}8491$ | $.0^{2}8266$ | $.0^{2}8046$ | $.0^{2}7832$ | $.0^{2}7623$ | $.0^{2}7418$ | $.0^{2}7219$ | $.0^{2}7024$ | $.0^{2}6835$ | $.0^{2}6649$ |
| 2.1 | $.0^{2}6468$ | $.0^{2}6292$ | $.0^{2}6120$ | $.0^{2}5952$ | $.0^{2}5788$ | $.0^{2}5628$ | $.0^{2}5472$ | $.0^{2}5320$ | $.0^{2}5172$ | $.0^{2}5028$ |
| 2.2 | $.0^{2}4887$ | $.0^{2}4750$ | $.0^{2}4616$ | $.0^{2}4486$ | $.0^{2}4358$ | $.0^{2}4235$ | $.0^{2}4114$ | $.0^{2}3996$ | $.0^{2}3882$ | $.0^{2}3770$ |
| 2.3 | $.0^{2}3662$ | $.0^{2}3556$ | $.0^{2}3453$ | $.0^{2}3352$ | $.0^{2}3255$ | $.0^{2}3159$ | $.0^{2}3067$ | $.0^{2}2977$ | $.0^{2}2889$ | $.0^{2}2804$ |
| 2.4 | $.0^{2}2720$ | $.0^{2}2640$ | $.0^{2}2561$ | $.0^{2}2484$ | $.0^{2}2410$ | $.0^{2}2337$ | $.0^{2}2267$ | $.0^{2}2199$ | $.0^{2}2132$ | $.0^{2}2067$ |
| 2.5 | $.0^{2}2004$ | $.0^{2}1943$ | $.0^{2}1883$ | $.0^{2}1826$ | $.0^{2}1769$ | $.0^{2}1715$ | $.0^{2}1662$ | $.0^{2}1610$ | $.0^{2}1560$ | $.0^{2}1511$ |
| 3.0 | $.0^{3}3822$ | $.0^{3}3689$ | $.0^{3}3560$ | $.0^{3}3436$ | $.0^{3}3316$ | $.0^{3}3199$ | $.0^{3}3087$ | $.0^{3}2978$ | $.0^{3}2873$ | $.0^{3}2771$ |
| 3.5 | $.0^{4}5848$ | $.0^{4}5620$ | $.0^{4}5400$ | $.0^{4}5188$ | $.0^{4}4984$ | $.0^{4}4788$ | $.0^{4}4599$ | $.0^{4}4417$ | $.0^{4}4242$ | $.0^{4}4073$ |
| 4.0 | $.0^{5}7145$ | $.0^{5}6835$ | $.0^{5}6538$ | $.0^{5}6253$ | $.0^{5}5980$ | $.0^{5}5718$ | $.0^{5}5468$ | $.0^{5}5227$ | $.0^{5}4997$ | $.0^{5}4777$ |

*Illustration:* The value of $L(D)$ for $D = 3.01$ is $.0^{3}3689 = .0003689$.

*Source:* Abridged from Table IV of Robert Schlaifer, *Introduction to Statistics for Business Decisions,* copyright 1961 by the President and Fellows of Harvard College, published by the McGraw-Hill Book Company, Inc., by permission of the author and the copyright holder.

## TABLE A-10 Five-Place Logarithms

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $-\infty$ | 00000 | 30103 | 47712 | 60206 | 69897 | 77815 | 84510 | 90309 | 95424 |
| 10 | 00000 | 00432 | 00860 | 01284 | 01703 | 02119 | 02531 | 02938 | 03342 | 03743 |
| 11 | 04139 | 04532 | 04922 | 05308 | 05690 | 06070 | 06446 | 06819 | 07188 | 07555 |
| 12 | 07918 | 08279 | 08636 | 08991 | 09342 | 09691 | 10037 | 10380 | 10721 | 11059 |
| 13 | 11394 | 11727 | 12057 | 12385 | 12710 | 13033 | 13354 | 13672 | 13988 | 14301 |
| 14 | 14613 | 14922 | 15229 | 15534 | 15836 | 16137 | 16435 | 16732 | 17026 | 17319 |
| 15 | 17609 | 17898 | 18184 | 18469 | 18752 | 19033 | 19312 | 19590 | 19866 | 20140 |
| 16 | 20412 | 20683 | 20952 | 21219 | 21484 | 21748 | 22011 | 22272 | 22531 | 22789 |
| 17 | 23045 | 23300 | 23533 | 23805 | 24055 | 24304 | 24551 | 24797 | 25042 | 25285 |
| 18 | 25527 | 25768 | 26007 | 26245 | 26482 | 26717 | 26951 | 27184 | 27416 | 27646 |
| 19 | 27875 | 28103 | 28330 | 28556 | 28780 | 29003 | 29226 | 29447 | 29667 | 29885 |
| 20 | 30103 | 30320 | 30535 | 30750 | 30963 | 31175 | 31387 | 31597 | 31806 | 32015 |
| 21 | 32222 | 32428 | 32634 | 32838 | 33041 | 33244 | 33445 | 33646 | 33846 | 34044 |
| 22 | 34242 | 34439 | 34635 | 34830 | 35025 | 35218 | 35411 | 35603 | 35793 | 35984 |
| 23 | 36173 | 36361 | 36549 | 36736 | 36922 | 37107 | 37291 | 37475 | 37658 | 37840 |
| 24 | 38021 | 38202 | 38382 | 38561 | 38739 | 38917 | 39094 | 39270 | 39445 | 39620 |
| 25 | 39794 | 39967 | 40140 | 40312 | 40483 | 40654 | 40824 | 40993 | 41162 | 41330 |
| 26 | 41497 | 41664 | 41830 | 41996 | 42160 | 42325 | 42488 | 42651 | 42813 | 42975 |
| 27 | 43136 | 43297 | 43457 | 43616 | 43775 | 43933 | 44091 | 44248 | 44404 | 44560 |
| 28 | 44716 | 44871 | 45025 | 45179 | 45332 | 45484 | 45637 | 45788 | 45939 | 46090 |
| 29 | 46240 | 46389 | 46538 | 46687 | 46835 | 46982 | 47129 | 47276 | 47422 | 47567 |
| 30 | 47712 | 47857 | 48001 | 48144 | 48287 | 48430 | 48572 | 48714 | 48855 | 48996 |
| 31 | 49136 | 49276 | 49415 | 49554 | 49693 | 49831 | 49969 | 50106 | 50243 | 50379 |
| 32 | 50515 | 50651 | 50786 | 50920 | 51055 | 51188 | 51322 | 51455 | 51587 | 51720 |
| 33 | 51851 | 51983 | 52114 | 52244 | 52375 | 52504 | 52634 | 52763 | 52892 | 53020 |
| 34 | 53148 | 53275 | 53403 | 53529 | 53656 | 53782 | 53908 | 54033 | 54158 | 54283 |
| 35 | 54407 | 54531 | 54654 | 54777 | 54900 | 55023 | 55145 | 55267 | 55388 | 55509 |
| 36 | 55630 | 55751 | 55871 | 55991 | 56110 | 56229 | 56348 | 56467 | 56585 | 56703 |
| 37 | 56820 | 56937 | 57054 | 57171 | 57287 | 57403 | 57519 | 57634 | 57749 | 57864 |
| 38 | 57978 | 58092 | 58206 | 58320 | 58433 | 58546 | 58659 | 58771 | 58883 | 58995 |
| 39 | 59106 | 59218 | 59329 | 59439 | 59550 | 59660 | 59770 | 59879 | 59988 | 60097 |
| 40 | 60206 | 60314 | 60423 | 60531 | 60638 | 60746 | 60853 | 60959 | 61066 | 61172 |
| 41 | 61278 | 61384 | 61490 | 61595 | 61700 | 61805 | 61909 | 62014 | 62118 | 62221 |
| 42 | 62325 | 62428 | 62531 | 62634 | 62737 | 62839 | 62941 | 63043 | 63144 | 63246 |
| 43 | 63347 | 63448 | 63548 | 63649 | 63749 | 63849 | 63949 | 64048 | 64147 | 64246 |
| 44 | 64345 | 64444 | 64542 | 64640 | 64738 | 64836 | 64933 | 65031 | 65128 | 65225 |
| 45 | 65321 | 65418 | 65514 | 65610 | 65706 | 65801 | 65896 | 65992 | 66087 | 66181 |
| 46 | 66276 | 66370 | 66464 | 66558 | 66652 | 66745 | 66839 | 66932 | 67025 | 67117 |
| 47 | 67210 | 67302 | 67394 | 67486 | 67578 | 67669 | 67761 | 67852 | 67943 | 68034 |
| 48 | 68124 | 68215 | 68305 | 68395 | 68485 | 68574 | 68664 | 68753 | 68842 | 68931 |
| 49 | 69020 | 69108 | 69197 | 69285 | 69373 | 69461 | 69548 | 69636 | 69723 | 69810 |

(*Continued*)

## TABLE A-10 Five-Place Logarithms (*Continued*)

| N | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 50 | 69897 | 69984 | 70070 | 70157 | 70243 | 70329 | 70415 | 70501 | 70586 | 70672 |
| 51 | 70757 | 70842 | 70927 | 71012 | 71096 | 71181 | 71265 | 71349 | 71433 | 71517 |
| 52 | 71600 | 71684 | 71767 | 71850 | 71933 | 72016 | 72099 | 72181 | 72263 | 72346 |
| 53 | 72428 | 72509 | 72591 | 72673 | 72754 | 72835 | 72916 | 72997 | 73078 | 73159 |
| 54 | 73239 | 73320 | 73400 | 73480 | 73560 | 73640 | 73719 | 73799 | 73878 | 73957 |
| 55 | 74036 | 74115 | 74194 | 74273 | 74351 | 74429 | 74507 | 74586 | 74663 | 74741 |
| 56 | 74819 | 74896 | 74974 | 75051 | 75128 | 75205 | 75282 | 75358 | 75435 | 75511 |
| 57 | 75587 | 75664 | 75740 | 75815 | 75891 | 75967 | 76042 | 76118 | 76193 | 76268 |
| 58 | 76343 | 76418 | 76492 | 76567 | 76641 | 76716 | 76790 | 76864 | 76938 | 77012 |
| 59 | 77085 | 77159 | 77232 | 77305 | 77379 | 77452 | 77525 | 77597 | 77670 | 77743 |
| 60 | 77815 | 77887 | 77960 | 78032 | 78104 | 78176 | 78247 | 78319 | 78390 | 78462 |
| 61 | 78533 | 78604 | 78675 | 78746 | 78817 | 78888 | 78958 | 79029 | 79099 | 79169 |
| 62 | 79239 | 79309 | 79379 | 79449 | 79518 | 79588 | 79657 | 79727 | 79796 | 79865 |
| 63 | 79934 | 80003 | 80072 | 80140 | 80209 | 80277 | 80346 | 80414 | 80482 | 80550 |
| 64 | 80618 | 80686 | 80754 | 80821 | 80889 | 80956 | 81023 | 81090 | 81158 | 81224 |
| 65 | 81291 | 81358 | 81425 | 81491 | 81558 | 81624 | 81690 | 81757 | 81823 | 81889 |
| 66 | 81954 | 82020 | 82086 | 82151 | 82217 | 82282 | 82347 | 82413 | 82478 | 82543 |
| 67 | 82607 | 82672 | 82737 | 82802 | 82866 | 82930 | 82995 | 83059 | 83123 | 83187 |
| 68 | 83251 | 83315 | 83378 | 83442 | 83506 | 83569 | 83632 | 83696 | 83759 | 83822 |
| 69 | 83885 | 83948 | 84011 | 84073 | 84136 | 84198 | 84261 | 84323 | 84386 | 84448 |
| 70 | 84510 | 84572 | 84634 | 84696 | 84757 | 84819 | 84880 | 84942 | 85003 | 85065 |
| 71 | 85126 | 85187 | 85248 | 85309 | 85370 | 85431 | 85491 | 85552 | 85612 | 85673 |
| 72 | 85733 | 85794 | 85854 | 85914 | 85974 | 86034 | 86094 | 86153 | 86213 | 86273 |
| 73 | 86332 | 86392 | 86451 | 86510 | 86570 | 86629 | 86688 | 86747 | 86806 | 86864 |
| 74 | 86923 | 86982 | 87040 | 87099 | 87157 | 87216 | 87274 | 87332 | 87390 | 87448 |
| 75 | 87506 | 87564 | 87622 | 87679 | 87737 | 87795 | 87852 | 87910 | 87967 | 88024 |
| 76 | 88081 | 88138 | 88195 | 88252 | 88309 | 88366 | 88423 | 88480 | 88536 | 88593 |
| 77 | 88649 | 88705 | 88762 | 88818 | 88874 | 88930 | 88986 | 89042 | 89098 | 89154 |
| 78 | 89209 | 89265 | 89321 | 89376 | 89432 | 89487 | 89542 | 89597 | 89653 | 89708 |
| 79 | 89763 | 89818 | 89873 | 89927 | 89982 | 90037 | 90091 | 90146 | 90200 | 90255 |
| 80 | 90309 | 90363 | 90417 | 90472 | 90526 | 90580 | 90634 | 90687 | 90741 | 90795 |
| 81 | 90849 | 90902 | 90956 | 91009 | 91062 | 91116 | 91169 | 91222 | 91275 | 91328 |
| 82 | 91381 | 91434 | 91487 | 91540 | 91593 | 91645 | 91698 | 91751 | 91803 | 91855 |
| 83 | 91908 | 91960 | 92012 | 92065 | 92117 | 92169 | 92221 | 92273 | 92324 | 92376 |
| 84 | 92428 | 92480 | 92531 | 92583 | 92634 | 92686 | 92737 | 92788 | 92840 | 92891 |
| 85 | 92942 | 92993 | 93044 | 93095 | 93146 | 93197 | 93247 | 93298 | 93349 | 93399 |
| 86 | 93450 | 93500 | 93551 | 93601 | 93651 | 93702 | 93752 | 93802 | 93852 | 93902 |
| 87 | 93952 | 94002 | 94052 | 94101 | 94151 | 94201 | 94250 | 94300 | 94349 | 94399 |
| 88 | 94448 | 94498 | 94547 | 94596 | 94645 | 94694 | 94743 | 94792 | 94841 | 94890 |
| 89 | 94939 | 94988 | 95036 | 95085 | 95134 | 95182 | 95231 | 95279 | 95328 | 95376 |
| 90 | 95424 | 95472 | 95521 | 95569 | 95617 | 95665 | 95713 | 95761 | 95809 | 95856 |
| 91 | 95904 | 95952 | 95999 | 96047 | 96095 | 96142 | 96190 | 96237 | 96284 | 96332 |
| 92 | 96379 | 96426 | 96473 | 96520 | 96567 | 96614 | 96661 | 96708 | 96755 | 96802 |
| 93 | 96848 | 96895 | 96942 | 96988 | 97035 | 97081 | 97128 | 97174 | 97220 | 97267 |
| 94 | 97313 | 97359 | 97405 | 97451 | 97497 | 97543 | 97589 | 97635 | 97681 | 97727 |
| 95 | 97772 | 97818 | 97864 | 97909 | 97955 | 98000 | 98046 | 98091 | 98137 | 98182 |
| 96 | 98227 | 98272 | 98318 | 98363 | 98408 | 98453 | 98498 | 98543 | 98588 | 98632 |
| 97 | 98677 | 98722 | 98767 | 98811 | 98856 | 98900 | 98945 | 98989 | 99034 | 99078 |
| 98 | 99123 | 99167 | 99211 | 99255 | 99300 | 99344 | 99388 | 99432 | 99476 | 99520 |
| 99 | 99564 | 99607 | 99651 | 99695 | 99739 | 99782 | 99826 | 99870 | 99913 | 99957 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ |
|---|---|---|---|---|
| 1 | 1 | 1.000 000 | 3.162 278 | 1.0000000 |
| 2 | 4 | 1.414 214 | 4.472 136 | .5000000 |
| 3 | 9 | 1.732 051 | 5.477 226 | .3333333 |
| 4 | 16 | 2.000 000 | 6.324 555 | .2500000 |
| 5 | 25 | 2.236 068 | 7.071 068 | .2000000 |
| 6 | 36 | 2.449 490 | 7.745 967 | .1666667 |
| 7 | 49 | 2.645 751 | 8.366 600 | .1428571 |
| 8 | 64 | 2.828 427 | 8.944 272 | .1250000 |
| 9 | 81 | 3.000 000 | 9.486 833 | .1111111 |
| 10 | 100 | 3.162 278 | 10.00000 | .1000000 |
| 11 | 121 | 3.316 625 | 10.48809 | .09090909 |
| 12 | 144 | 3.464 102 | 10.95445 | .08333333 |
| 13 | 169 | 3.605 551 | 11.40175 | .07692308 |
| 14 | 196 | 3.741 657 | 11.83216 | .07142857 |
| 15 | 225 | 3.872 983 | 12.24745 | .06666667 |
| 16 | 256 | 4.000 000 | 12.64911 | .06250000 |
| 17 | 289 | 4.123 106 | 13.03840 | .05882353 |
| 18 | 324 | 4.242 641 | 13.41641 | .05555556 |
| 19 | 361 | 4.358 899 | 13.78405 | .05263158 |
| 20 | 400 | 4.472 136 | 14.14214 | .05000000 |
| 21 | 441 | 4.582 576 | 14.49138 | .04761905 |
| 22 | 484 | 4.690 416 | 14.83240 | .04545455 |
| 23 | 529 | 4.795 832 | 15.16575 | .04347826 |
| 24 | 576 | 4.898 979 | 15.49193 | .04166667 |
| 25 | 625 | 5.000 000 | 15.81139 | .04000000 |
| 26 | 676 | 5.099 020 | 16.12452 | .03846154 |
| 27 | 729 | 5.196 152 | 16.43168 | .03703704 |
| 28 | 784 | 5.291 503 | 16.73320 | .03571429 |
| 29 | 841 | 5.385 165 | 17.02939 | .03448276 |
| 30 | 900 | 5.477 226 | 17.32051 | .03333333 |
| 31 | 961 | 5.567 874 | 17.60682 | .03225806 |
| 32 | 1 024 | 5.656 854 | 17.88854 | .03125000 |
| 33 | 1 089 | 5.744 563 | 18.16590 | .03030303 |
| 34 | 1 156 | 5.830 952 | 18.43909 | .02941176 |
| 35 | 1 225 | 5.916 080 | 18.70829 | .02857143 |
| 36 | 1 296 | 6.000 000 | 18.97367 | .02777778 |
| 37 | 1 369 | 6.082 763 | 19.23538 | .02702703 |
| 38 | 1 444 | 6.164 414 | 19.49359 | .02631579 |
| 39 | 1 521 | 6.244 998 | 19.74842 | .02564103 |
| 40 | 1 600 | 6.324 555 | 20.00000 | .02500000 |
| 41 | 1 681 | 6.403 124 | 20.24846 | .02439024 |
| 42 | 1 764 | 6.480 741 | 20.49390 | .02380952 |
| 43 | 1 849 | 6.557 439 | 20.73644 | .02325581 |
| 44 | 1 936 | 6.633 250 | 20.97618 | .02272727 |
| 45 | 2 025 | 6.708 204 | 21.21320 | .02222222 |
| 46 | 2 116 | 6.782 330 | 21.44761 | .02173913 |
| 47 | 2 209 | 6.855 655 | 21.67948 | .02127660 |
| 48 | 2 304 | 6.928 203 | 21.90890 | .02083333 |
| 49 | 2 401 | 7.000 000 | 22.13594 | .02040816 |
| 50 | 2 500 | 7.071 068 | 22.36068 | .02000000 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .0 |
|---|---|---|---|---|
| 50 | 2 500 | 7.071 068 | 22.36068 | 2000000 |
| 51 | 2 601 | 7.141 428 | 22.58318 | 1960784 |
| 52 | 2 704 | 7.211 103 | 22.80351 | 1923077 |
| 53 | 2 809 | 7.280 110 | 23.02173 | 1886792 |
| 54 | 2 916 | 7.348 469 | 23.23790 | 1851852 |
| 55 | 3 025 | 7.416 198 | 23.45208 | 1818182 |
| 56 | 3 136 | 7.483 315 | 23.66432 | 1785714 |
| 57 | 3 249 | 7.549 834 | 23.87467 | 1754386 |
| 58 | 3 364 | 7.615 773 | 24.08319 | 1724138 |
| 59 | 3 481 | 7.681 146 | 24.28992 | 1694915 |
| 60 | 3 600 | 7.745 967 | 24.49490 | 1666667 |
| 61 | 3 721 | 7.810 250 | 24.69818 | 1639344 |
| 62 | 3 844 | 7.874 008 | 24.89980 | 1612903 |
| 63 | 3 969 | 7.937 254 | 25.09980 | 1587302 |
| 64 | 4 096 | 8.000 000 | 25.29822 | 1562500 |
| 65 | 4 225 | 8.062 258 | 25.49510 | 1538462 |
| 66 | 4 356 | 8.124 038 | 25.69047 | 1515152 |
| 67 | 4 489 | 8.185 353 | 25.88436 | 1492537 |
| 68 | 4 624 | 8.246 211 | 26.07681 | 1470588 |
| 69 | 4 761 | 8.306 624 | 26.26785 | 1449275 |
| 70 | 4 900 | 8.366 600 | 26.45751 | 1428571 |
| 71 | 5 041 | 8.426 150 | 26.64583 | 1408451 |
| 72 | 5 184 | 8.485 281 | 26.83282 | 1388889 |
| 73 | 5 329 | 8.544 004 | 27.01851 | 1369863 |
| 74 | 5 476 | 8.602 325 | 27.20294 | 1351351 |
| 75 | 5 625 | 8.660 254 | 27.38613 | 1333333 |
| 76 | 5 776 | 8.717 798 | 27.56810 | 1315789 |
| 77 | 5 929 | 8.774 964 | 27.74887 | 1298701 |
| 78 | 6 084 | 8.831 761 | 27.92848 | 1282051 |
| 79 | 6 241 | 8.888 194 | 28.10694 | 1265823 |
| 80 | 6 400 | 8.944 272 | 28.28427 | 1250000 |
| 81 | 6 561 | 9.000 000 | 28.46050 | 1234568 |
| 82 | 6 724 | 9.055 385 | 28.63564 | 1219512 |
| 83 | 6 889 | 9.110 434 | 28.80972 | 1204819 |
| 84 | 7 056 | 9.165 151 | 28.98275 | 1190476 |
| 85 | 7 225 | 9.219 544 | 29.15476 | 1176471 |
| 86 | 7 396 | 9.273 618 | 29.32576 | 1162791 |
| 87 | 7 569 | 9.327 379 | 29.49576 | 1149425 |
| 88 | 7 744 | 9.380 832 | 29.66479 | 1136364 |
| 89 | 7 921 | 9.433 981 | 29.83287 | 1123596 |
| 90 | 8 100 | 9.486 833 | 30.00000 | 1111111 |
| 91 | 8 281 | 9.539 392 | 30.16621 | 1098901 |
| 92 | 8 464 | 9.591 663 | 30.33150 | 1086957 |
| 93 | 8 649 | 9.643 651 | 30.49590 | 1075269 |
| 94 | 8 836 | 9.695 360 | 30.65942 | 1063830 |
| 95 | 9 025 | 9.746 794 | 30.82207 | 1052632 |
| 96 | 9 216 | 9.797 959 | 30.98387 | 1041667 |
| 97 | 9 409 | 9.848 858 | 31.14482 | 1030928 |
| 98 | 9 604 | 9.899 495 | 31.30495 | 1020408 |
| 99 | 9 801 | 9.949 874 | 31.46427 | 1010101 |
| 100 | 10 000 | 10.00000 | 31.62278 | 1000000 |

(*Continued*)

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $\frac{1/N}{.0}$ |
|---|---|---|---|---|
| 100 | 10 000 | 10.00000 | 31.62278 | 10000000 |
| 101 | 10 201 | 10.04988 | 31.78050 | 09900990 |
| 102 | 10 404 | 10.09950 | 31.93744 | 09803922 |
| 103 | 10 609 | 10.14889 | 32.09361 | 09708738 |
| 104 | 10 816 | 10.19804 | 32.24903 | 09615385 |
| 105 | 11 025 | 10.24695 | 32.40370 | 09523810 |
| 106 | 11 236 | 10.29563 | 32.55764 | 09433962 |
| 107 | 11 449 | 10.34408 | 32.71085 | 09345794 |
| 108 | 11 664 | 10.39230 | 32.86335 | 09259259 |
| 109 | 11 881 | 10.44031 | 33.01515 | 09174312 |
| 110 | 12 100 | 10.48809 | 33.16625 | 09090909 |
| 111 | 12 321 | 10.53565 | 33.31666 | 09009009 |
| 112 | 12 544 | 10.58301 | 33.46640 | 08928571 |
| 113 | 12 769 | 10.63015 | 33.61547 | 08849558 |
| 114 | 12 996 | 10.67708 | 33.76389 | 08771930 |
| 115 | 13 225 | 10.72381 | 33.91165 | 08695652 |
| 116 | 13 456 | 10.77033 | 34.05877 | 08620690 |
| 117 | 13 689 | 10.81665 | 34.20526 | 08547009 |
| 118 | 13 924 | 10.86278 | 34.35113 | 08474576 |
| 119 | 14 161 | 10.90871 | 34.49638 | 08403361 |
| 120 | 14 400 | 10.95445 | 34.64102 | 08333333 |
| 121 | 14 641 | 11.00000 | 34.78505 | 08264463 |
| 122 | 14 884 | 11.04536 | 34.93850 | 08196721 |
| 123 | 15 129 | 11.09054 | 35.07136 | 08130081 |
| 124 | 15 376 | 11.13553 | 35.21363 | 08064516 |
| 125 | 15 625 | 11.18034 | 35.35534 | 08000000 |
| 126 | 15 876 | 11.22497 | 35.49648 | 07936508 |
| 127 | 16 129 | 11.26943 | 35.63706 | 07874016 |
| 128 | 16 384 | 11.31371 | 35.77709 | 07812500 |
| 129 | 16 641 | 11.35782 | 35.91657 | 07751938 |
| 130 | 16 900 | 11.40175 | 36.05551 | 07692308 |
| 131 | 17 161 | 11.44552 | 36.19392 | 07633588 |
| 132 | 17 424 | 11.48913 | 36.33180 | 07575758 |
| 133 | 17 689 | 11.53256 | 36.46917 | 07518797 |
| 134 | 17 956 | 11.57584 | 36.60601 | 07462687 |
| 135 | 18 225 | 11.61895 | 36.74235 | 07407407 |
| 136 | 18 496 | 11.66190 | 36.87818 | 07352941 |
| 137 | 18 769 | 11.70470 | 37.01351 | 07299270 |
| 138 | 19 044 | 11.74734 | 37.14835 | 07246377 |
| 139 | 19 321 | 11.78983 | 37.28270 | 07194245 |
| 140 | 19 600 | 11.83216 | 37.41657 | 07142857 |
| 141 | 19 881 | 11.87434 | 37.54997 | 07092199 |
| 142 | 20 164 | 11.91638 | 37.68289 | 07042254 |
| 143 | 20 449 | 11.95826 | 37.81534 | 06993007 |
| 144 | 20 736 | 12.00000 | 37.94733 | 06944444 |
| 145 | 21 025 | 12.04159 | 38.07887 | 06896552 |
| 146 | 21 316 | 12.08305 | 38.20995 | 06849315 |
| 147 | 21 609 | 12.12436 | 38.34058 | 06802721 |
| 148 | 21 904 | 12.16553 | 38.47077 | 06756757 |
| 149 | 22 201 | 12.20656 | 38.60052 | 06711409 |
| 150 | 22 500 | 12.24745 | 38.72983 | 06666667 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $\frac{1/N}{.00}$ |
|---|---|---|---|---|
| 150 | 22 500 | 12.24745 | 38.72983 | 6666667 |
| 151 | 22 801 | 12.28821 | 38.85872 | 6622517 |
| 152 | 23 104 | 12.32883 | 38.98718 | 6578947 |
| 153 | 23 409 | 12.36932 | 39.11521 | 6535948 |
| 154 | 23 716 | 12.40967 | 39.24283 | 6493506 |
| 155 | 24 025 | 12.44990 | 39.37004 | 6451613 |
| 156 | 24 336 | 12.49000 | 39.49684 | 6410256 |
| 157 | 24 649 | 12.52996 | 39.62323 | 6369427 |
| 158 | 24 964 | 12.56981 | 39.74921 | 6329114 |
| 159 | 25 281 | 12.60952 | 39.87480 | 6289308 |
| 160 | 25 600 | 12.64911 | 40.00000 | 6250000 |
| 161 | 25 921 | 12.68858 | 40.12481 | 6211180 |
| 162 | 26 244 | 12.72792 | 40.24922 | 6172840 |
| 163 | 26 569 | 12.76715 | 40.37326 | 6134969 |
| 164 | 26 896 | 12.80625 | 40.49691 | 6097561 |
| 165 | 27 225 | 12.84523 | 40.62019 | 6060606 |
| 166 | 27 556 | 12.88410 | 40.74310 | 6024096 |
| 167 | 27 889 | 12.92285 | 40.86563 | 5988024 |
| 168 | 28 224 | 12.96148 | 40.98780 | 5952381 |
| 169 | 28 561 | 13.00000 | 41.10961 | 5917160 |
| 170 | 28 900 | 13.03840 | 41.23106 | 5882353 |
| 171 | 29 241 | 13.07670 | 41.35215 | 5847953 |
| 172 | 29 584 | 13.11488 | 41.47288 | 5813953 |
| 173 | 29 929 | 13.15295 | 41.59327 | 5780347 |
| 174 | 30 276 | 13.19091 | 41.71331 | 5747126 |
| 175 | 30 625 | 13.22876 | 41.83300 | 5714286 |
| 176 | 30 976 | 13.26650 | 41.95235 | 5681818 |
| 177 | 31 329 | 13.30413 | 42.07137 | 5649718 |
| 178 | 31 684 | 13.34166 | 42.19005 | 5617978 |
| 179 | 32 041 | 13.37909 | 42.30839 | 5586592 |
| 180 | 32 400 | 13.41641 | 42.42641 | 5555556 |
| 181 | 32 761 | 13.45362 | 42.54409 | 5524862 |
| 182 | 33 124 | 13.49074 | 42.66146 | 5494505 |
| 183 | 33 489 | 13.52775 | 42.77850 | 5464481 |
| 184 | 33 856 | 13.56466 | 42.89522 | 5434783 |
| 185 | 34 225 | 13.60147 | 43.01163 | 5405405 |
| 186 | 34 596 | 13.63818 | 43.12772 | 5376344 |
| 187 | 34 969 | 13.67479 | 43.24350 | 5347594 |
| 188 | 35 344 | 13.71131 | 43.35897 | 5319149 |
| 189 | 35 721 | 13.74773 | 43.47413 | 5291005 |
| 190 | 36 100 | 13.78405 | 43.58899 | 5263158 |
| 191 | 36 481 | 13.82027 | 43.70355 | 5235602 |
| 192 | 36 864 | 13.85641 | 43.81780 | 5208333 |
| 193 | 37 249 | 13.89244 | 43.93177 | 5181347 |
| 194 | 37 636 | 13.92839 | 44.04543 | 5154639 |
| 195 | 38 025 | 13.96424 | 44.15880 | 5128205 |
| 196 | 38 416 | 14.00000 | 44.27189 | 5102041 |
| 197 | 38 809 | 14.03567 | 44.38468 | 5076142 |
| 198 | 39 204 | 14.07125 | 44.49719 | 5050505 |
| 199 | 39 601 | 14.10674 | 44.60942 | 5025126 |
| 200 | 40 000 | 14.14214 | 44.72136 | 5000000 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $\frac{1/N}{.00}$ |
|---|---|---|---|---|
| 200 | 40 000 | 14.14214 | 44.72136 | 5000000 |
| 201 | 40 401 | 14.17745 | 44.83302 | 4975124 |
| 202 | 40 804 | 14.21267 | 44.94441 | 4950495 |
| 203 | 41 209 | 14.24781 | 45.05552 | 4926108 |
| 204 | 41 616 | 14.28286 | 45.16636 | 4901961 |
| 205 | 42 025 | 14.31782 | 45.27693 | 4878049 |
| 206 | 42 436 | 14.35270 | 45.38722 | 4854369 |
| 207 | 42 849 | 14.38749 | 45.49725 | 4830918 |
| 208 | 43 264 | 14.42221 | 45.60702 | 4807692 |
| 209 | 43 681 | 14.45683 | 45.71652 | 4784689 |
| 210 | 44 100 | 14.49138 | 45.82576 | 4761905 |
| 211 | 44 521 | 14.52584 | 45.93474 | 4739336 |
| 212 | 44 944 | 14.56022 | 46.04346 | 4716981 |
| 213 | 45 369 | 14.59452 | 46.15192 | 4694836 |
| 214 | 45 796 | 14.62874 | 46.26013 | 4672897 |
| 215 | 46 225 | 14.66288 | 46.36809 | 4651163 |
| 216 | 46 656 | 14.69694 | 46.47580 | 4629630 |
| 217 | 47 089 | 14.73092 | 46.58326 | 4608295 |
| 218 | 47 524 | 14.76482 | 46.69047 | 4587156 |
| 219 | 47 961 | 14.79865 | 46.79744 | 4566210 |
| 220 | 48 400 | 14.83240 | 46.90416 | 4545455 |
| 221 | 48 841 | 14.86607 | 47.01064 | 4524887 |
| 222 | 49 284 | 14.89966 | 47.11688 | 4504505 |
| 223 | 49 729 | 14.93318 | 47.22288 | 4484305 |
| 224 | 50 176 | 14.96663 | 47.32864 | 4464286 |
| 225 | 50 625 | 15.00000 | 47.43416 | 4444444 |
| 226 | 51 076 | 15.03330 | 47.53946 | 4424779 |
| 227 | 51 529 | 15.06652 | 47.64452 | 4405286 |
| 228 | 51 984 | 15.09967 | 47.74935 | 4385965 |
| 229 | 52 441 | 15.13275 | 47.85394 | 4366812 |
| 230 | 52 900 | 15.16575 | 47.95832 | 4347826 |
| 231 | 53 361 | 15.19868 | 48.06246 | 4329004 |
| 232 | 53 824 | 15.23155 | 48.16638 | 4310345 |
| 233 | 54 289 | 15.26434 | 48.27007 | 4291845 |
| 234 | 54 756 | 15.29706 | 48.37355 | 4273504 |
| 235 | 55 225 | 15.32971 | 48.47680 | 4255319 |
| 236 | 55 696 | 15.36229 | 48.57983 | 4237288 |
| 237 | 56 169 | 15.39480 | 48.68265 | 4219409 |
| 238 | 56 644 | 15.42725 | 48.78524 | 4201681 |
| 239 | 57 121 | 15.45962 | 48.88763 | 4184100 |
| 240 | 57 600 | 15.49193 | 48.98979 | 4166667 |
| 241 | 58 081 | 15.52417 | 49.09175 | 4149378 |
| 242 | 58 564 | 15.55635 | 49.19350 | 4132231 |
| 243 | 59 049 | 15.58846 | 49.29503 | 4115226 |
| 244 | 59 536 | 15.62050 | 49.39636 | 4098361 |
| 245 | 60 025 | 15.65248 | 49.49747 | 4081633 |
| 246 | 60 516 | 15.68439 | 49.59839 | 4065041 |
| 247 | 61 009 | 15.71623 | 49.69909 | 4048583 |
| 248 | 61 504 | 15.74802 | 49.79960 | 4032258 |
| 249 | 62 001 | 15.77973 | 49.89990 | 4016064 |
| 250 | 62 500 | 15.81139 | 50.00000 | 4000000 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $\frac{1/N}{.00}$ |
|---|---|---|---|---|
| 250 | 62 500 | 15.81139 | 50.00000 | 4000000 |
| 251 | 63 001 | 15.84298 | 50.09990 | 3984064 |
| 252 | 63 504 | 15.87451 | 50.19960 | 3968254 |
| 253 | 64 009 | 15.90597 | 50.29911 | 3952569 |
| 254 | 64 516 | 15.93738 | 50.39841 | 3937008 |
| 255 | 65 025 | 15.96872 | 50.49752 | 3921569 |
| 256 | 65 536 | 16.00000 | 50.59644 | 3906250 |
| 257 | 66 049 | 16.03122 | 50.69517 | 3891051 |
| 258 | 66 564 | 16.06238 | 50.79370 | 3875969 |
| 259 | 67 081 | 16.09348 | 50.89204 | 3861004 |
| 260 | 67 600 | 16.12452 | 50.99020 | 3846154 |
| 261 | 68 121 | 16.15549 | 51.08816 | 3831418 |
| 262 | 68 644 | 16.18641 | 51.18594 | 3816794 |
| 263 | 69 169 | 16.21727 | 51.28353 | 3802281 |
| 264 | 69 696 | 16.24808 | 51.38093 | 3787879 |
| 265 | 70 225 | 16.27882 | 51.47815 | 3773585 |
| 266 | 70 756 | 16.30951 | 51.57519 | 3759398 |
| 267 | 71 289 | 16.34013 | 51.67204 | 3745318 |
| 268 | 71 824 | 16.37071 | 51.76872 | 3731343 |
| 269 | 72 361 | 16.40122 | 51.86521 | 3717472 |
| 270 | 72 900 | 16.43168 | 51.96152 | 3703704 |
| 271 | 73 441 | 16.46208 | 52.05766 | 3690037 |
| 272 | 73 984 | 16.49242 | 52.15362 | 3676471 |
| 273 | 74 529 | 16.52271 | 52.24940 | 3663004 |
| 274 | 75 076 | 16.55295 | 52.34501 | 3649635 |
| 275 | 75 625 | 16.58312 | 52.44044 | 3636364 |
| 276 | 76 176 | 16.61325 | 52.53570 | 3623188 |
| 277 | 76 729 | 16.64332 | 52.63079 | 3610108 |
| 278 | 77 284 | 16.67333 | 52.72571 | 3597122 |
| 279 | 77 841 | 16.70329 | 52.82045 | 3584229 |
| 280 | 78 400 | 16.73320 | 52.91503 | 3571429 |
| 281 | 78 961 | 16.76305 | 53.00943 | 3558719 |
| 282 | 79 524 | 16.79286 | 53.10367 | 3546099 |
| 283 | 80 089 | 16.82260 | 53.19774 | 3533569 |
| 284 | 80 656 | 16.85230 | 53.29165 | 3521127 |
| 285 | 81 225 | 16.88194 | 53.38539 | 3508772 |
| 286 | 81 796 | 16.91153 | 53.47897 | 3496503 |
| 287 | 82 369 | 16.94107 | 53.57238 | 3484321 |
| 288 | 82 944 | 16.97056 | 53.66563 | 3472222 |
| 289 | 83 521 | 17.00000 | 53.75872 | 3460208 |
| 290 | 84 100 | 17.02939 | 53.85165 | 3448276 |
| 291 | 84 681 | 17.05872 | 53.94442 | 3436426 |
| 292 | 85 264 | 17.08801 | 54.03702 | 3424658 |
| 293 | 85 849 | 17.11724 | 54.12947 | 3412969 |
| 294 | 86 436 | 17.14643 | 54.22177 | 3401361 |
| 295 | 87 025 | 17.17556 | 54.31390 | 3389831 |
| 296 | 87 616 | 17.20465 | 54.40588 | 3378378 |
| 297 | 88 209 | 17.23369 | 54.49771 | 3367003 |
| 298 | 88 804 | 17.26268 | 54.58938 | 3355705 |
| 299 | 89 401 | 17.29162 | 54.68089 | 3344482 |
| 300 | 90 000 | 17.32051 | 54.77226 | 3333333 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 300 | 90 000 | 17.32051 | 54.77226 | 3333333 |
| 301 | 90 601 | 17.34935 | 54.86347 | 3322259 |
| 302 | 91 204 | 17.37815 | 54.95453 | 3311258 |
| 303 | 91 809 | 17.40690 | 55.04544 | 3300330 |
| 304 | 92 416 | 17.43560 | 55.13620 | 3289474 |
| 305 | 93 025 | 17.46425 | 55.22681 | 3278689 |
| 306 | 93 636 | 17.49286 | 55.31727 | 3267974 |
| 307 | 94 249 | 17.52142 | 55.40758 | 3257329 |
| 308 | 94 864 | 17.54993 | 55.49775 | 3246753 |
| 309 | 95 481 | 17.57840 | 55.58777 | 3236246 |
| 310 | 96 100 | 17.60682 | 55.67764 | 3225806 |
| 311 | 96 721 | 17.63519 | 55.76737 | 3215434 |
| 312 | 97 344 | 17.66352 | 55.85696 | 3205128 |
| 313 | 97 969 | 17.69181 | 55.94640 | 3194888 |
| 314 | 98 596 | 17.72005 | 56.03570 | 3184713 |
| 315 | 99 225 | 17.74824 | 56.12486 | 3174603 |
| 316 | 99 856 | 17.77639 | 56.21388 | 3164557 |
| 317 | 100 489 | 17.80449 | 56.30275 | 3154574 |
| 318 | 101 124 | 17.83255 | 56.39149 | 3144654 |
| 319 | 101 761 | 17.86057 | 56.48008 | 3134796 |
| 320 | 102 400 | 17.88854 | 56.56854 | 3125000 |
| 321 | 103 041 | 17.91647 | 56.65686 | 3115265 |
| 322 | 103 684 | 17.94436 | 56.74504 | 3105590 |
| 323 | 104 329 | 17.97220 | 56.83309 | 3095975 |
| 324 | 104 976 | 18.00000 | 56.92100 | 3086420 |
| 325 | 105 625 | 18.02776 | 57.00877 | 3076923 |
| 326 | 106 276 | 18.05547 | 57.09641 | 3067485 |
| 327 | 106 929 | 18.08314 | 57.18391 | 3058104 |
| 328 | 107 584 | 18.11077 | 57.27128 | 3048780 |
| 329 | 108 241 | 18.13836 | 57.35852 | 3039514 |
| 330 | 108 900 | 18.16590 | 57.44563 | 3030303 |
| 331 | 109 561 | 18.19341 | 57.53260 | 3021148 |
| 332 | 110 224 | 18.22087 | 57.61944 | 3012048 |
| 333 | 110 889 | 18.24829 | 57.70615 | 3003003 |
| 334 | 111 556 | 18.27567 | 57.79273 | 2994012 |
| 335 | 112 225 | 18.30301 | 57.87918 | 2985075 |
| 336 | 112 896 | 18.33030 | 57.96551 | 2976190 |
| 337 | 113 569 | 18.35756 | 58.05170 | 2967359 |
| 338 | 114 244 | 18.38478 | 58.13777 | 2958580 |
| 339 | 114 921 | 18.41195 | 58.22371 | 2949853 |
| 340 | 115 600 | 18.43909 | 58.30952 | 2941176 |
| 341 | 116 281 | 18.46619 | 58.39521 | 2932551 |
| 342 | 116 964 | 18.49324 | 58.48077 | 2923977 |
| 343 | 117 649 | 18.52026 | 58.56620 | 2915452 |
| 344 | 118 336 | 18.54724 | 58.65151 | 2906977 |
| 345 | 119 025 | 18.57418 | 58.73670 | 2898551 |
| 346 | 119 716 | 18.60108 | 58.82176 | 2890173 |
| 347 | 120 409 | 18.62794 | 58.90671 | 2881844 |
| 348 | 121 104 | 18.65476 | 58.99152 | 2873563 |
| 349 | 121 801 | 18.68154 | 59.07622 | 2865330 |
| 350 | 122 500 | 18.70829 | 59.16080 | 2857143 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 350 | 122 500 | 18.70829 | 59.16080 | 2857143 |
| 351 | 123 201 | 18.73499 | 59.24525 | 2849003 |
| 352 | 123 904 | 18.76166 | 59.32959 | 2840909 |
| 353 | 124 609 | 18.78829 | 59.41380 | 2832861 |
| 354 | 125 316 | 18.81489 | 59.49790 | 2824859 |
| 355 | 126 025 | 18.84144 | 59.58188 | 2816901 |
| 356 | 126 736 | 18.86796 | 59.66574 | 2808989 |
| 357 | 127 449 | 18.89444 | 59.74948 | 2801120 |
| 358 | 128 164 | 18.92089 | 59.83310 | 2793296 |
| 359 | 128 881 | 18.94730 | 59.91661 | 2785515 |
| 360 | 129 600 | 18.97367 | 60.00000 | 2777778 |
| 361 | 130 321 | 19.00000 | 60.08328 | 2770083 |
| 362 | 131 044 | 19.02630 | 60.16644 | 2762431 |
| 363 | 131 769 | 19.05256 | 60.24948 | 2754821 |
| 364 | 132 496 | 19.07878 | 60.33241 | 2747253 |
| 365 | 133 225 | 19.10497 | 60.41523 | 2739726 |
| 366 | 133 956 | 19.13113 | 60.49793 | 2732240 |
| 367 | 134 689 | 19.15724 | 60.58052 | 2724796 |
| 368 | 135 424 | 19.18333 | 60.66300 | 2717391 |
| 369 | 136 161 | 19.20937 | 60.74537 | 2710027 |
| 370 | 136 900 | 19.23538 | 60.82763 | 2702703 |
| 371 | 137 641 | 19.26136 | 60.90977 | 2695418 |
| 372 | 138 384 | 19.28730 | 60.99180 | 2688172 |
| 373 | 139 129 | 19.31321 | 61.07373 | 2680965 |
| 374 | 139 876 | 19.33908 | 61.15554 | 2673797 |
| 375 | 140 625 | 19.36492 | 61.23724 | 2666667 |
| 376 | 141 376 | 19.39072 | 61.31884 | 2659574 |
| 377 | 142 129 | 19.41649 | 61.40033 | 2652520 |
| 378 | 142 884 | 19.44222 | 61.48170 | 2645503 |
| 379 | 143 641 | 19.46792 | 61.56298 | 2638522 |
| 380 | 144 400 | 19.49359 | 61.64414 | 2631579 |
| 381 | 145 161 | 19.51922 | 61.72520 | 2624672 |
| 382 | 145 924 | 19.54483 | 61.80615 | 2617801 |
| 383 | 146 689 | 19.57039 | 61.88699 | 2610966 |
| 384 | 147 456 | 19.59592 | 61.96773 | 2604167 |
| 385 | 148 225 | 19.62142 | 62.04837 | 2597403 |
| 386 | 148 996 | 19.64688 | 62.12890 | 2590674 |
| 387 | 149 769 | 19.67232 | 62.20932 | 2583979 |
| 388 | 150 544 | 19.69772 | 62.28965 | 2577320 |
| 389 | 151 321 | 19.72308 | 62.36986 | 2570694 |
| 390 | 152 100 | 19.74842 | 62.44998 | 2564103 |
| 391 | 152 881 | 19.77372 | 62.52999 | 2557545 |
| 392 | 153 664 | 19.79899 | 62.60990 | 2551020 |
| 393 | 154 449 | 19.82423 | 62.68971 | 2544529 |
| 394 | 155 236 | 19.84943 | 62.76942 | 2538071 |
| 395 | 156 025 | 19.87461 | 62.84903 | 2531646 |
| 396 | 156 816 | 19.89975 | 62.92853 | 2525253 |
| 397 | 157 609 | 19.92486 | 63.00794 | 2518892 |
| 398 | 158 404 | 19.94994 | 63.08724 | 2512563 |
| 399 | 159 201 | 19.97498 | 63.16645 | 2506266 |
| 400 | 160 000 | 20.00000 | 63.24555 | 2500000 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 400 | 160 000 | 20.00000 | 63.24555 | 2500000 |
| 401 | 160 801 | 20.02498 | 63.32456 | 2493766 |
| 402 | 161 604 | 20.04994 | 63.40347 | 2487562 |
| 403 | 162 409 | 20.07486 | 63.48228 | 2481390 |
| 404 | 163 216 | 20.09975 | 63.56099 | 2475248 |
| 405 | 164 025 | 20.12461 | 63.63961 | 2469136 |
| 406 | 164 836 | 20.14944 | 63.71813 | 2463054 |
| 407 | 165 649 | 20.17424 | 63.79655 | 2457002 |
| 408 | 166 464 | 20.19901 | 63.87488 | 2450980 |
| 409 | 167 281 | 20.22375 | 63.95311 | 2444988 |
| 410 | 168 100 | 20.24846 | 64.03124 | 2439024 |
| 411 | 168 921 | 20.27313 | 64.10928 | 2433090 |
| 412 | 169 744 | 20.29778 | 64.18723 | 2427184 |
| 413 | 170 569 | 20.32240 | 64.26508 | 2421308 |
| 414 | 171 396 | 20.34699 | 64.34283 | 2415459 |
| 415 | 172 225 | 20.37155 | 64.42049 | 2409639 |
| 416 | 173 056 | 20.39608 | 64.49806 | 2403846 |
| 417 | 173 889 | 20.42058 | 64.57554 | 2398082 |
| 418 | 174 724 | 20.44505 | 64.65292 | 2392344 |
| 419 | 175 561 | 20.46949 | 64.73021 | 2386635 |
| 420 | 176 400 | 20.49390 | 64.80741 | 2380952 |
| 421 | 177 241 | 20.51828 | 64.88451 | 2375297 |
| 422 | 178 084 | 20.54264 | 64.96153 | 2369668 |
| 423 | 178 929 | 20.56696 | 65.03845 | 2364066 |
| 424 | 179 776 | 20.59126 | 65.11528 | 2358491 |
| 425 | 180 625 | 20.61553 | 65.19202 | 2352941 |
| 426 | 181 476 | 20.63977 | 65.26868 | 2347418 |
| 427 | 182 329 | 20.66398 | 65.34524 | 2341920 |
| 428 | 183 184 | 20.68816 | 65.42171 | 2336449 |
| 429 | 184 041 | 20.71232 | 65.49809 | 2331002 |
| 430 | 184 900 | 20.73644 | 65.57439 | 2325581 |
| 431 | 185 761 | 20.76054 | 65.65059 | 2320186 |
| 432 | 186 624 | 20.78461 | 65.72671 | 2314815 |
| 433 | 187 489 | 20.80865 | 65.80274 | 2309469 |
| 434 | 188 356 | 20.83267 | 65.87868 | 2304147 |
| 435 | 189 225 | 20.85665 | 65.95453 | 2298851 |
| 436 | 190 096 | 20.88061 | 66.03030 | 2293578 |
| 437 | 190 969 | 20.90454 | 66.10598 | 2288330 |
| 438 | 191 844 | 20.92845 | 66.18157 | 2283105 |
| 439 | 192 721 | 20.95233 | 66.25708 | 2277904 |
| 440 | 193 600 | 20.97618 | 66.33250 | 2272727 |
| 441 | 194 481 | 21.00000 | 66.40783 | 2267574 |
| 442 | 195 364 | 21.02380 | 66.48308 | 2262443 |
| 443 | 196 249 | 21.04757 | 66.55825 | 2257336 |
| 444 | 197 136 | 21.07131 | 66.63332 | 2252252 |
| 445 | 198 025 | 21.09502 | 66.70832 | 2247191 |
| 446 | 198 916 | 21.11871 | 66.78323 | 2242152 |
| 447 | 199 809 | 21.14237 | 66.85806 | 2237136 |
| 448 | 200 704 | 21.16601 | 66.93280 | 2232143 |
| 449 | 201 601 | 21.18962 | 67.00746 | 2227171 |
| 450 | 202 500 | 21.21320 | 67.08204 | 2222222 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 450 | 202 500 | 21.21320 | 67.08204 | 2222222 |
| 451 | 203 401 | 21.23676 | 67.15653 | 2217295 |
| 452 | 204 304 | 21.26029 | 67.23095 | 2212389 |
| 453 | 205 209 | 21.28380 | 67.30537 | 2207506 |
| 454 | 206 116 | 21.30728 | 67.37952 | 2202643 |
| 455 | 207 025 | 21.33073 | 67.45369 | 2197802 |
| 456 | 207 936 | 21.35416 | 67.52777 | 2192982 |
| 457 | 208 849 | 21.37756 | 67.60178 | 2188184 |
| 458 | 209 764 | 21.40093 | 67.67570 | 2183406 |
| 459 | 210 681 | 21.42429 | 67.74954 | 2178649 |
| 460 | 211 600 | 21.44761 | 67.82330 | 2173913 |
| 461 | 212 521 | 21.47091 | 67.89698 | 2169197 |
| 462 | 213 444 | 21.49419 | 67.97058 | 2164502 |
| 463 | 214 369 | 21.41743 | 68.04410 | 2159827 |
| 464 | 215 296 | 21.54066 | 68.11755 | 2155172 |
| 465 | 216 225 | 21.56386 | 68.19091 | 2150538 |
| 466 | 217 156 | 21.58703 | 68.26419 | 2145923 |
| 467 | 218 089 | 21.61018 | 68.33740 | 2141328 |
| 468 | 219 024 | 21.63331 | 68.41053 | 2136752 |
| 469 | 219 961 | 21.65641 | 68.48357 | 2132196 |
| 470 | 220 900 | 21.67948 | 68.55655 | 2127660 |
| 471 | 221 841 | 21.70253 | 68.62944 | 2123142 |
| 472 | 222 784 | 21.72556 | 68.70226 | 2118644 |
| 473 | 223 729 | 21.74856 | 68.77500 | 2114165 |
| 474 | 224 676 | 21.77154 | 68.84766 | 2109705 |
| 475 | 225 625 | 21.79449 | 68.92024 | 2105263 |
| 476 | 226 576 | 21.81742 | 68.99275 | 2100840 |
| 477 | 227 529 | 21.84033 | 69.06519 | 2096436 |
| 478 | 228 484 | 21.86321 | 69.13754 | 2092050 |
| 479 | 229 441 | 21.88607 | 69.20983 | 2087683 |
| 480 | 230 400 | 21.90890 | 69.28203 | 2083333 |
| 481 | 231 361 | 21.93171 | 69.35416 | 2079002 |
| 482 | 232 324 | 21.95450 | 69.42622 | 2074689 |
| 483 | 233 289 | 21.97726 | 69.49820 | 2070393 |
| 484 | 234 256 | 22.00000 | 69.57011 | 2066116 |
| 485 | 235 225 | 22.02272 | 69.64194 | 2061856 |
| 486 | 236 196 | 22.04541 | 69.71370 | 2057613 |
| 487 | 237 169 | 22.06808 | 69.78539 | 2053388 |
| 488 | 238 144 | 22.09072 | 69.85700 | 2049180 |
| 489 | 239 121 | 22.11334 | 69.92853 | 2044990 |
| 490 | 240 100 | 22.13594 | 70.00000 | 2040816 |
| 491 | 241 081 | 22.15852 | 70.07139 | 2036660 |
| 492 | 242 064 | 22.18107 | 70.14271 | 2032520 |
| 493 | 243 049 | 22.20360 | 70.21396 | 2028398 |
| 494 | 244 036 | 22.22611 | 70.28513 | 2024291 |
| 495 | 245 025 | 22.24860 | 70.35624 | 2020202 |
| 496 | 246 016 | 22.27106 | 70.42727 | 2016129 |
| 497 | 247 009 | 22.29350 | 70.49823 | 2012072 |
| 498 | 248 004 | 22.31591 | 70.56912 | 2008032 |
| 499 | 249 001 | 22.33831 | 70.63993 | 2004008 |
| 500 | 250 000 | 22.36068 | 70.71068 | 2000000 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 500 | 250 000 | 22.36068 | 70.71068 | 2000000 |
| 501 | 251 001 | 22.38303 | 70.78135 | 1996008 |
| 502 | 252 004 | 22.40536 | 70.85196 | 1992032 |
| 503 | 253 009 | 22.42766 | 70.92249 | 1988072 |
| 504 | 254 016 | 22.44994 | 70.99296 | 1984127 |
| 505 | 255 025 | 22.47221 | 71.06335 | 1980198 |
| 506 | 256 036 | 22.49444 | 71.13368 | 1976285 |
| 507 | 257 049 | 22.51666 | 71.20393 | 1972387 |
| 508 | 258 064 | 22.53886 | 71.27412 | 1968504 |
| 509 | 259 081 | 22.56103 | 71.34424 | 1964637 |
| 510 | 260 100 | 22.58318 | 71.41428 | 1960784 |
| 511 | 261 121 | 22.60531 | 71.48426 | 1956947 |
| 512 | 262 144 | 22.62742 | 71.55418 | 1953125 |
| 513 | 263 169 | 22.64950 | 71.62402 | 1949318 |
| 514 | 264 196 | 22.67157 | 71.69379 | 1945525 |
| 515 | 265 225 | 22.69361 | 71.76350 | 1941748 |
| 516 | 266 256 | 22.71563 | 71.83314 | 1937984 |
| 517 | 267 289 | 22.73763 | 71.90271 | 1934236 |
| 518 | 268 324 | 22.75961 | 71.97222 | 1930502 |
| 519 | 269 361 | 22.78157 | 72.04165 | 1926782 |
| 520 | 270 400 | 22.80351 | 72.11103 | 1923077 |
| 521 | 271 441 | 22.82542 | 72.18033 | 1919386 |
| 522 | 272 484 | 22.84732 | 72.24957 | 1915709 |
| 523 | 273 529 | 22.86919 | 72.31874 | 1912046 |
| 524 | 274 576 | 22.89105 | 72.38784 | 1908397 |
| 525 | 275 625 | 22.91288 | 72.45688 | 1904762 |
| 526 | 276 676 | 22.93469 | 72.52586 | 1901141 |
| 527 | 277 729 | 22.95648 | 72.59477 | 1897533 |
| 528 | 278 784 | 22.97825 | 72.66361 | 1893939 |
| 529 | 279 841 | 23.00000 | 72.73239 | 1890359 |
| 530 | 280 900 | 23.02173 | 72.80110 | 1886792 |
| 531 | 281 961 | 23.04344 | 72.86975 | 1883239 |
| 532 | 283 024 | 23.06513 | 72.93833 | 1879699 |
| 533 | 284 089 | 23.08679 | 73.00685 | 1876173 |
| 534 | 285 156 | 23.10844 | 73.07530 | 1872659 |
| 535 | 286 225 | 23.13007 | 73.14369 | 1869159 |
| 536 | 287 296 | 23.15167 | 73.21202 | 1865672 |
| 537 | 288 369 | 23.17326 | 73.28028 | 1862197 |
| 538 | 289 444 | 23.19483 | 73.34848 | 1858736 |
| 539 | 290 521 | 23.21637 | 73.41662 | 1855288 |
| 540 | 291 600 | 23.23790 | 73.48469 | 1851852 |
| 541 | 292 681 | 23.25941 | 73.55270 | 1848429 |
| 542 | 293 764 | 23.28089 | 73.62065 | 1845018 |
| 543 | 294 849 | 23.30236 | 73.68853 | 1841621 |
| 544 | 295 936 | 23.32381 | 73.75636 | 1838235 |
| 545 | 297 025 | 23.34524 | 73.82412 | 1834862 |
| 546 | 298 116 | 23.36664 | 73.89181 | 1831502 |
| 547 | 299 209 | 23.38803 | 73.95945 | 1828154 |
| 548 | 300 304 | 23.40940 | 74.02702 | 1824818 |
| 549 | 301 401 | 23.43075 | 74.09453 | 1821494 |
| 550 | 302 500 | 23.45208 | 74.16198 | 1818182 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 550 | 302 500 | 23.45208 | 74.16198 | 1818182 |
| 551 | 303 601 | 23.47339 | 74.22937 | 1814882 |
| 552 | 304 704 | 23.49468 | 74.29670 | 1811594 |
| 553 | 305 809 | 23.51595 | 74.36397 | 1808318 |
| 554 | 306 916 | 23.53720 | 74.43118 | 1805054 |
| 555 | 308 025 | 23.55844 | 74.49832 | 1801802 |
| 556 | 309 136 | 23.57965 | 74.56541 | 1798561 |
| 557 | 310 249 | 23.60085 | 74.63243 | 1795332 |
| 558 | 311 364 | 23.62202 | 74.69940 | 1792115 |
| 559 | 312 481 | 23.64318 | 74.76630 | 1788909 |
| 560 | 313 600 | 23.66432 | 74.83315 | 1785714 |
| 561 | 314 721 | 23.68544 | 74.89993 | 1782531 |
| 562 | 315 844 | 23.70654 | 74.96666 | 1779359 |
| 563 | 316 969 | 23.72762 | 75.03333 | 1776199 |
| 564 | 318 096 | 23.74868 | 75.09993 | 1773050 |
| 565 | 319 225 | 23.76973 | 75.16648 | 1769912 |
| 566 | 320 356 | 23.79075 | 75.23297 | 1766784 |
| 567 | 321 489 | 23.81176 | 75.29940 | 1763668 |
| 568 | 322 624 | 23.83275 | 75.36577 | 1760563 |
| 569 | 323 761 | 23.85372 | 75.43209 | 1757469 |
| 570 | 324 900 | 23.87467 | 75.49834 | 1754386 |
| 571 | 326 041 | 23.89561 | 75.56454 | 1751313 |
| 572 | 327 184 | 23.91652 | 75.63068 | 1748252 |
| 573 | 328 329 | 23.93742 | 75.69676 | 1745201 |
| 574 | 329 476 | 23.95830 | 75.76279 | 1742160 |
| 575 | 330 625 | 23.97916 | 75.82875 | 1739130 |
| 576 | 331 776 | 24.00000 | 75.89466 | 1736111 |
| 577 | 332 929 | 24.02082 | 75.96052 | 1733102 |
| 578 | 334 084 | 24.04163 | 76.02631 | 1730104 |
| 579 | 335 241 | 24.06242 | 76.09205 | 1727116 |
| 580 | 336 400 | 24.08319 | 76.15773 | 1724138 |
| 581 | 337 561 | 24.10394 | 76.22336 | 1721170 |
| 582 | 338 724 | 24.12468 | 76.28892 | 1718213 |
| 583 | 339 889 | 24.14539 | 76.35444 | 1715266 |
| 584 | 341 056 | 24.16609 | 76.41989 | 1712329 |
| 585 | 342 225 | 24.18677 | 76.48529 | 1709402 |
| 586 | 343 396 | 24.20744 | 76.55064 | 1706485 |
| 587 | 344 569 | 24.22808 | 76.61593 | 1703578 |
| 588 | 345 744 | 24.24871 | 76.68116 | 1700680 |
| 589 | 346 921 | 24.26932 | 76.74634 | 1697793 |
| 590 | 348 100 | 24.28992 | 76.81146 | 1694915 |
| 591 | 349 281 | 24.31049 | 76.87652 | 1692047 |
| 592 | 350 464 | 24.33105 | 76.94154 | 1689189 |
| 593 | 351 649 | 24.35159 | 77.00649 | 1686341 |
| 594 | 352 836 | 24.37212 | 77.07140 | 1683502 |
| 595 | 354 025 | 24.39262 | 77.13624 | 1680672 |
| 596 | 355 216 | 24.41311 | 77.20104 | 1677852 |
| 597 | 356 409 | 24.43358 | 77.26578 | 1675042 |
| 598 | 357 604 | 24.45404 | 77.33046 | 1672241 |
| 599 | 358 801 | 24.47448 | 77.39509 | 1669449 |
| 600 | 360 000 | 24.49490 | 77.45967 | 1666667 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $\frac{1/N}{.00}$ |
|---|---|---|---|---|
| 600 | 360 000 | 24.49490 | 77.45967 | 1666667 |
| 601 | 361 201 | 24.51530 | 77.52419 | 1663894 |
| 602 | 362 404 | 24.53569 | 77.58866 | 1661130 |
| 603 | 363 609 | 24.55606 | 77.65307 | 1658375 |
| 604 | 364 816 | 24.57641 | 77.71744 | 1655629 |
| 605 | 366 025 | 24.59675 | 77.78165 | 1652893 |
| 606 | 367 236 | 24.61707 | 77.84600 | 1650165 |
| 607 | 368 449 | 24.63737 | 77.91020 | 1647446 |
| 608 | 369 664 | 24.65766 | 77.97435 | 1644737 |
| 609 | 370 881 | 24.67793 | 78.03845 | 1642036 |
| 610 | 372 100 | 24.69818 | 78.10250 | 1639344 |
| 611 | 373 321 | 24.71841 | 78.16649 | 1636661 |
| 612 | 374 544 | 24.73863 | 78.23043 | 1633987 |
| 613 | 375 769 | 24.75884 | 78.29432 | 1631321 |
| 614 | 376 996 | 24.77902 | 78.35815 | 1628664 |
| 615 | 378 225 | 24.79919 | 78.42194 | 1626016 |
| 616 | 379 456 | 24.81935 | 78.48567 | 1623377 |
| 617 | 380 689 | 24.83948 | 78.54935 | 1620746 |
| 618 | 381 924 | 24.85961 | 78.61298 | 1618123 |
| 619 | 383 161 | 24.87971 | 78.67655 | 1615509 |
| 620 | 384 400 | 24.89980 | 78.74008 | 1612903 |
| 621 | 385 641 | 24.91987 | 78.80355 | 1610306 |
| 622 | 386 884 | 24.93993 | 78.86698 | 1607717 |
| 623 | 388 129 | 24.95997 | 78.93035 | 1605136 |
| 624 | 389 376 | 24.97999 | 78.99367 | 1602564 |
| 625 | 390 625 | 25.00000 | 79.05694 | 1600000 |
| 626 | 391 876 | 25.01999 | 79.12016 | 1597444 |
| 627 | 393 129 | 25.03997 | 79.18333 | 1594896 |
| 628 | 394 384 | 25.05993 | 79.24645 | 1592357 |
| 629 | 395 641 | 25.07987 | 79.30952 | 1589825 |
| 630 | 396 900 | 25.09980 | 79.37254 | 1587302 |
| 631 | 398 161 | 25.11971 | 79.43551 | 1584786 |
| 632 | 399 424 | 25.13961 | 79.49843 | 1582278 |
| 633 | 400 689 | 25.15949 | 79.56130 | 1579779 |
| 634 | 401 956 | 25.17936 | 79.62412 | 1577287 |
| 635 | 403 225 | 25.19921 | 79.68689 | 1574803 |
| 636 | 404 496 | 25.21904 | 79.74961 | 1572327 |
| 637 | 405 769 | 25.23886 | 79.81228 | 1569859 |
| 638 | 407 044 | 25.25866 | 79.87490 | 1567398 |
| 639 | 408 321 | 25.27845 | 79.93748 | 1564945 |
| 640 | 409 600 | 25.29822 | 80.00000 | 1562500 |
| 641 | 410 881 | 25.31798 | 80.06248 | 1560062 |
| 642 | 412 164 | 25.33772 | 80.12490 | 1557632 |
| 643 | 413 449 | 25.35744 | 80.18728 | 1555210 |
| 644 | 414 736 | 25.37716 | 80.24961 | 1552795 |
| 645 | 416 025 | 25.39685 | 80.31189 | 1550388 |
| 646 | 417 316 | 25.41653 | 80.37413 | 1547988 |
| 647 | 418 609 | 25.43619 | 80.43631 | 1545595 |
| 648 | 419 904 | 25.45584 | 80.49845 | 1543210 |
| 649 | 421 201 | 25.47548 | 80.56054 | 1540832 |
| 650 | 422 500 | 25.49510 | 80.62258 | 1538462 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $\frac{1/N}{.00}$ |
|---|---|---|---|---|
| 650 | 422 500 | 25.49510 | 80.62258 | 1538462 |
| 651 | 423 801 | 25.51470 | 80.68457 | 1536098 |
| 652 | 425 104 | 25.53429 | 80.74652 | 1533742 |
| 653 | 426 409 | 25.55386 | 80.80842 | 1531394 |
| 654 | 427 716 | 25.57342 | 80.87027 | 1529052 |
| 655 | 429 025 | 25.59297 | 80.93207 | 1526718 |
| 656 | 430 336 | 25.61250 | 80.99383 | 1524390 |
| 657 | 431 649 | 25.63201 | 81.05554 | 1522070 |
| 658 | 432 964 | 25.65141 | 81.11720 | 1519757 |
| 659 | 434 281 | 25.67100 | 81.17881 | 1517451 |
| 660 | 435 600 | 25.69047 | 81.24038 | 1515152 |
| 661 | 436 921 | 25.70992 | 81.30191 | 1512859 |
| 662 | 438 244 | 25.72936 | 81.36338 | 1510574 |
| 663 | 439 569 | 25.74879 | 81.42481 | 1508296 |
| 664 | 440 896 | 25.76820 | 81.48620 | 1506024 |
| 665 | 442 225 | 25.78759 | 81.54753 | 1503759 |
| 666 | 443 556 | 25.80698 | 81.60882 | 1501502 |
| 667 | 444 889 | 25.82634 | 81.67007 | 1499250 |
| 668 | 446 224 | 25.84570 | 81.73127 | 1497006 |
| 669 | 447 561 | 25.86503 | 81.79242 | 1494768 |
| 670 | 448 900 | 25.88436 | 81.85353 | 1492537 |
| 671 | 450 241 | 25.90367 | 81.91459 | 1490313 |
| 672 | 451 584 | 25.92296 | 81.97561 | 1488095 |
| 673 | 452 929 | 25.94224 | 82.03658 | 1485884 |
| 674 | 454 276 | 25.96151 | 82.09750 | 1483680 |
| 675 | 455 625 | 25.98076 | 82.15838 | 1481481 |
| 676 | 456 976 | 26.00000 | 82.21922 | 1479290 |
| 677 | 458 329 | 26.01922 | 82.28001 | 1477105 |
| 678 | 459 684 | 26.03843 | 82.34076 | 1474926 |
| 679 | 461 041 | 26.05763 | 82.40146 | 1472754 |
| 680 | 462 400 | 26.07681 | 82.46211 | 1470588 |
| 681 | 463 761 | 26.09598 | 82.42272 | 1468429 |
| 682 | 465 124 | 26.11513 | 82.58329 | 1466276 |
| 683 | 466 489 | 26.13427 | 82.64381 | 1464129 |
| 684 | 467 856 | 26.15339 | 82.70429 | 1461988 |
| 685 | 469 225 | 26.17250 | 82.76473 | 1459854 |
| 686 | 470 596 | 26.19160 | 82.82512 | 1457726 |
| 687 | 471 969 | 26.21068 | 82.88546 | 1455604 |
| 688 | 473 344 | 26.22975 | 82.94577 | 1453488 |
| 689 | 474 721 | 26.24881 | 83.00602 | 1451379 |
| 690 | 476 100 | 26.26785 | 83.06624 | 1449275 |
| 691 | 477 481 | 26.28688 | 83.12641 | 1447178 |
| 692 | 478 864 | 26.30589 | 83.18654 | 1445087 |
| 693 | 480 249 | 26.32489 | 83.24662 | 1443001 |
| 694 | 481 636 | 26.34388 | 83.30666 | 1440922 |
| 695 | 483 025 | 26.36285 | 83.36666 | 1438849 |
| 696 | 484 416 | 26.38181 | 83.42661 | 1436782 |
| 697 | 485 809 | 26.40076 | 83.48653 | 1434720 |
| 698 | 487 204 | 26.41969 | 83.54639 | 1432665 |
| 699 | 488 601 | 26.43861 | 83.60622 | 1430615 |
| 700 | 490 000 | 26.45751 | 83.66600 | 1428571 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 700 | 490 000 | 26.45751 | 83.66600 | 1428571 |
| 701 | 491 401 | 26.47640 | 83.72574 | 1426534 |
| 702 | 492 804 | 26.49528 | 83.78544 | 1424501 |
| 703 | 494 209 | 26.51415 | 83.84510 | 1422475 |
| 704 | 495 616 | 26.53300 | 83.90471 | 1420455 |
| 705 | 497 025 | 26.55184 | 83.96428 | 1418440 |
| 706 | 498 436 | 26.57066 | 84.02381 | 1416431 |
| 707 | 499 849 | 26.58947 | 84.08329 | 1414427 |
| 708 | 501 264 | 26.60827 | 84.14274 | 1412429 |
| 709 | 502 681 | 26.62705 | 84.20214 | 1410437 |
| 710 | 504 100 | 26.64583 | 84.26150 | 1408451 |
| 711 | 505 521 | 26.66458 | 84.32082 | 1406470 |
| 712 | 506 944 | 26.68333 | 84.38009 | 1404494 |
| 713 | 508 369 | 26.70206 | 84.43933 | 1402525 |
| 714 | 509 796 | 26.72078 | 84.49852 | 1400560 |
| 715 | 511 225 | 26.73948 | 84.55767 | 1398601 |
| 716 | 512 656 | 26.75818 | 84.61678 | 1396648 |
| 717 | 514 089 | 26.77686 | 84.67585 | 1394700 |
| 718 | 515 524 | 26.79552 | 84.73488 | 1392758 |
| 719 | 516 961 | 26.81418 | 84.79387 | 1390821 |
| 720 | 518 400 | 26.83282 | 84.85281 | 1388889 |
| 721 | 519 841 | 26.85144 | 84.91172 | 1386963 |
| 722 | 521 284 | 26.87006 | 84.97058 | 1385042 |
| 723 | 522 729 | 26.88866 | 85.02941 | 1383126 |
| 724 | 524 176 | 26.90725 | 85.08819 | 1381215 |
| 725 | 525 625 | 26.92582 | 85.14693 | 1379310 |
| 726 | 527 076 | 26.94439 | 85.20563 | 1377410 |
| 727 | 528 529 | 26.96294 | 85.26429 | 1375516 |
| 728 | 529 984 | 26.98148 | 85.32292 | 1373626 |
| 729 | 531 441 | 27.00000 | 85.38150 | 1371742 |
| 730 | 532 900 | 27.01851 | 85.44004 | 1369863 |
| 731 | 534 361 | 27.03701 | 85.49854 | 1367989 |
| 732 | 535 824 | 27.05550 | 85.55700 | 1366120 |
| 733 | 537 289 | 27.07397 | 85.61542 | 1364256 |
| 734 | 538 756 | 27.09243 | 85.67380 | 1372398 |
| 735 | 540 225 | 27.11088 | 85.73214 | 1360544 |
| 736 | 541 696 | 27.12932 | 85.79044 | 1358696 |
| 737 | 543 169 | 27.14774 | 85.84870 | 1356852 |
| 738 | 544 644 | 27.16616 | 85.90693 | 1355014 |
| 739 | 546 121 | 27.18455 | 85.96511 | 1353180 |
| 740 | 547 600 | 27.20294 | 86.02325 | 1351351 |
| 741 | 549 081 | 27.22132 | 86.08136 | 1349528 |
| 742 | 550 564 | 27.23968 | 86.13942 | 1347709 |
| 743 | 552 049 | 27.25803 | 86.19745 | 1345895 |
| 744 | 553 536 | 27.27636 | 86.25543 | 1344086 |
| 745 | 555 025 | 27.29469 | 86.31338 | 1342282 |
| 746 | 556 516 | 27.31300 | 86.37129 | 1340483 |
| 747 | 558 009 | 27.33130 | 86.42916 | 1338688 |
| 748 | 559 504 | 27.34959 | 86.48699 | 1336898 |
| 749 | 561 001 | 27.36786 | 86.54479 | 1335113 |
| 750 | 562 500 | 27.38613 | 86.60254 | 1333333 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 750 | 562 500 | 27.38613 | 86.60254 | 1333333 |
| 751 | 564 001 | 27.40438 | 86.66026 | 1331558 |
| 752 | 565 504 | 27.42262 | 86.71793 | 1329787 |
| 753 | 567 009 | 27.44085 | 86.77557 | 1328021 |
| 754 | 568 516 | 27.45906 | 86.83317 | 1326260 |
| 755 | 570 025 | 27.47726 | 86.89074 | 1324503 |
| 756 | 571 536 | 27.49545 | 86.94826 | 1322751 |
| 757 | 573 049 | 27.51363 | 87.00575 | 1321004 |
| 758 | 574 564 | 27.53180 | 87.06320 | 1319261 |
| 759 | 576 081 | 27.54995 | 87.12061 | 1317523 |
| 760 | 577 600 | 27.56810 | 87.17798 | 1315789 |
| 761 | 579 121 | 27.58623 | 87.23531 | 1314060 |
| 762 | 580 644 | 27.60435 | 87.29261 | 1312336 |
| 763 | 582 169 | 27.62245 | 87.34987 | 1310616 |
| 764 | 583 696 | 27.64055 | 87.40709 | 1308901 |
| 765 | 585 225 | 27.65863 | 87.46428 | 1307190 |
| 766 | 586 756 | 27.67671 | 87.52143 | 1305483 |
| 767 | 588 289 | 27.69476 | 87.57854 | 1303781 |
| 768 | 589 824 | 27.71281 | 87.63561 | 1302083 |
| 769 | 591 361 | 27.73085 | 87.69265 | 1300390 |
| 770 | 592 900 | 27.74887 | 87.74964 | 1298701 |
| 771 | 594 441 | 27.76689 | 87.80661 | 1297017 |
| 772 | 595 984 | 27.78489 | 87.86353 | 1295337 |
| 773 | 597 529 | 27.80288 | 87.92042 | 1293661 |
| 774 | 599 076 | 27.82086 | 87.97727 | 1291990 |
| 775 | 600 625 | 27.83882 | 88.03408 | 1290323 |
| 776 | 602 176 | 27.85678 | 88.09086 | 1288660 |
| 777 | 603 729 | 27.87472 | 88.14760 | 1287001 |
| 778 | 605 284 | 27.89265 | 88.20431 | 1285347 |
| 779 | 606 841 | 27.91057 | 88.26098 | 1283697 |
| 780 | 608 400 | 27.92848 | 88.31761 | 1282051 |
| 781 | 609 961 | 27.94638 | 88.37420 | 1280410 |
| 782 | 611 524 | 27.96426 | 88.43076 | 1278772 |
| 783 | 613 089 | 27.98214 | 88.48729 | 1277139 |
| 784 | 614 656 | 28.00000 | 88.54377 | 1275510 |
| 785 | 616 225 | 28.01785 | 88.60023 | 1273885 |
| 786 | 617 796 | 28.03569 | 88.65664 | 1272265 |
| 787 | 619 369 | 28.05352 | 88.71302 | 1270648 |
| 788 | 620 944 | 28.07134 | 88.76936 | 1269036 |
| 789 | 622 521 | 28.08914 | 88.82567 | 1267427 |
| 790 | 624 100 | 28.10694 | 88.88194 | 1265823 |
| 791 | 625 681 | 28.12472 | 88.93818 | 1264223 |
| 792 | 627 264 | 28.14249 | 88.99438 | 1262626 |
| 793 | 628 849 | 28.16026 | 89.05055 | 1261034 |
| 794 | 630 436 | 28.17801 | 89.10668 | 1259446 |
| 795 | 632 025 | 28.19574 | 89.16277 | 1257862 |
| 796 | 633 616 | 28.21347 | 89.21883 | 1256281 |
| 797 | 635 209 | 28.23119 | 89.27486 | 1254705 |
| 798 | 636 804 | 28.24889 | 89.33085 | 1253133 |
| 799 | 638 401 | 28.26659 | 89.38680 | 1251564 |
| 800 | 640 000 | 28.28427 | 89.44272 | 1250000 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 800 | 640 000 | 28.28427 | 89.44272 | 1250000 |
| 801 | 641 601 | 28.30194 | 89.49860 | 1248439 |
| 802 | 643 204 | 28.31960 | 89.55445 | 1246883 |
| 803 | 644 809 | 28.33725 | 89.61027 | 1245330 |
| 804 | 646 416 | 28.35489 | 89.66605 | 1243781 |
| 805 | 648 025 | 28.37252 | 89.72179 | 1242236 |
| 806 | 649 636 | 28.39014 | 89.77750 | 1240695 |
| 807 | 651 249 | 28.40775 | 89.83318 | 1239157 |
| 808 | 652 864 | 28.42534 | 89.88882 | 1237624 |
| 809 | 654 481 | 28.44293 | 89.94443 | 1236094 |
| 810 | 656 100 | 28.46050 | 90.00000 | 1234568 |
| 811 | 657 721 | 28.47806 | 90.05554 | 1233046 |
| 812 | 659 344 | 28.49561 | 90.11104 | 1231527 |
| 813 | 660 969 | 28.51315 | 90.16651 | 1230012 |
| 814 | 662 596 | 28.53069 | 90.22195 | 1228501 |
| 815 | 664 225 | 28.54820 | 90.27735 | 1226994 |
| 816 | 665 856 | 28.56571 | 90.33272 | 1225490 |
| 817 | 667 489 | 28.58321 | 90.38805 | 1223990 |
| 818 | 669 124 | 28.60070 | 90.44335 | 1222494 |
| 819 | 670 761 | 28.61818 | 90.49862 | 1221001 |
| 820 | 672 400 | 28.63564 | 90.55385 | 1219512 |
| 821 | 674 041 | 28.65310 | 90.60905 | 1218027 |
| 822 | 675 684 | 28.67054 | 90.66422 | 1216545 |
| 823 | 677 329 | 28.68798 | 90.71935 | 1215067 |
| 824 | 678 976 | 28.70540 | 90.77445 | 1213592 |
| 825 | 680 625 | 28.72281 | 90.82951 | 1212121 |
| 826 | 682 276 | 28.74022 | 90.88454 | 1210654 |
| 827 | 683 929 | 28.75761 | 90.93954 | 1209190 |
| 828 | 685 584 | 28.77499 | 90.99451 | 1207729 |
| 829 | 687 241 | 28.79236 | 91.04944 | 1206273 |
| 830 | 688 900 | 28.80972 | 91.10434 | 1204819 |
| 831 | 690 561 | 28.82707 | 91.15920 | 1203369 |
| 832 | 692 224 | 28.84441 | 91.21403 | 1201923 |
| 833 | 693 889 | 28.86174 | 91.26883 | 1200480 |
| 834 | 695 556 | 28.87906 | 91.32360 | 1199041 |
| 835 | 697 225 | 28.89637 | 91.37833 | 1197605 |
| 836 | 698 896 | 28.91366 | 91.43304 | 1196172 |
| 837 | 700 569 | 28.93095 | 91.48770 | 1194743 |
| 838 | 702 244 | 28.94823 | 91.54234 | 1193317 |
| 839 | 703 921 | 28.96550 | 91.59694 | 1191895 |
| 840 | 705 600 | 28.98275 | 91.65151 | 1190476 |
| 841 | 707 281 | 29.00000 | 91.70605 | 1189061 |
| 842 | 708 964 | 29.01724 | 91.76056 | 1187648 |
| 843 | 710 649 | 29.03446 | 91.81503 | 1186240 |
| 844 | 712 336 | 29.05168 | 91.86947 | 1184834 |
| 845 | 714 025 | 29.06888 | 91.92388 | 1183432 |
| 846 | 715 716 | 29.08608 | 91.97826 | 1182033 |
| 847 | 717 409 | 29.10326 | 92.03260 | 1180638 |
| 848 | 719 104 | 29.12044 | 92.08692 | 1179245 |
| 849 | 720 801 | 29.13760 | 92.14120 | 1177856 |
| 850 | 722 500 | 29.15476 | 92.19544 | 1176471 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 850 | 722 500 | 29.15476 | 92.19544 | 1176471 |
| 851 | 724 201 | 29.17190 | 92.24966 | 1175088 |
| 852 | 725 904 | 29.18904 | 92.30385 | 1173709 |
| 853 | 727 609 | 29.20616 | 92.35800 | 1172333 |
| 854 | 729 316 | 29.22328 | 92.41212 | 1170960 |
| 855 | 731 025 | 29.24038 | 92.46621 | 1169591 |
| 856 | 732 736 | 29.25748 | 92.52027 | 1168224 |
| 857 | 734 449 | 29.27456 | 92.57429 | 1166861 |
| 858 | 736 164 | 29.29164 | 92.62829 | 1165501 |
| 859 | 737 881 | 29.30870 | 92.68225 | 1164144 |
| 860 | 739 600 | 29.32576 | 92.73618 | 1162791 |
| 861 | 741 321 | 29.34280 | 92.79009 | 1161440 |
| 862 | 743 044 | 29.35984 | 92.84396 | 1160093 |
| 863 | 744 769 | 29.37686 | 92.89779 | 1158749 |
| 864 | 746 496 | 29.39388 | 92.95160 | 1157407 |
| 865 | 748 225 | 29.41088 | 93.00538 | 1156069 |
| 866 | 749 956 | 29.42788 | 93.05912 | 1154734 |
| 867 | 751 689 | 29.44486 | 93.11283 | 1153403 |
| 868 | 753 424 | 29.46184 | 93.16652 | 1152074 |
| 869 | 755 161 | 29.47881 | 93.22017 | 1150748 |
| 870 | 756 900 | 29.49576 | 93.27379 | 1149425 |
| 871 | 758 641 | 29.51271 | 93.32738 | 1148106 |
| 872 | 760 384 | 29.52965 | 93.38094 | 1146789 |
| 873 | 762 129 | 29.54657 | 93.43447 | 1145475 |
| 874 | 763 876 | 29.56349 | 93.48797 | 1144165 |
| 875 | 765 625 | 29.58040 | 93.54143 | 1142857 |
| 876 | 767 376 | 29.59730 | 93.59487 | 1141553 |
| 877 | 769 129 | 29.61419 | 93.64828 | 1140251 |
| 878 | 770 884 | 29.63106 | 93.70165 | 1138952 |
| 879 | 772 641 | 29.64793 | 93.75500 | 1137656 |
| 880 | 774 400 | 29.66479 | 93.80832 | 1136364 |
| 881 | 776 161 | 29.68164 | 93.86160 | 1135074 |
| 882 | 777 924 | 29.69848 | 93.91486 | 1133787 |
| 883 | 779 689 | 29.71532 | 93.96808 | 1132503 |
| 884 | 781 456 | 29.73214 | 94.02127 | 1131222 |
| 885 | 783 225 | 29.74895 | 94.07444 | 1129944 |
| 886 | 784 996 | 29.76575 | 94.12757 | 1128668 |
| 887 | 786 769 | 29.78255 | 94.18068 | 1127396 |
| 888 | 788 544 | 29.79933 | 94.23375 | 1126126 |
| 889 | 790 321 | 29.81610 | 94.28680 | 1124859 |
| 890 | 792 100 | 29.83287 | 94.33981 | 1123596 |
| 891 | 793 881 | 29.84962 | 94.39280 | 1122334 |
| 892 | 795 664 | 29.86637 | 94.44575 | 1121076 |
| 893 | 797 449 | 29.88311 | 94.49868 | 1119821 |
| 894 | 799 236 | 29.89983 | 94.55157 | 1118568 |
| 895 | 801 025 | 29.91655 | 94.60444 | 1117318 |
| 896 | 802 816 | 29.93326 | 94.65728 | 1116071 |
| 897 | 804 609 | 29.94996 | 94.71008 | 1114827 |
| 898 | 806 404 | 29.96665 | 94.76286 | 1113586 |
| 899 | 808 201 | 29.98333 | 94.81561 | 1112347 |
| 900 | 810 000 | 30.00000 | 94.86833 | 1111111 |

## TABLE A-11 Squares, Square Roots, and Reciprocals 1–1000 (*Continued*)

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 900 | 810 000 | 30.00000 | 94.86833 | 1111111 |
| 901 | 811 801 | 30.01666 | 94.92102 | 1109878 |
| 902 | 813 604 | 30.03331 | 94.97368 | 1108647 |
| 903 | 815 409 | 30.04996 | 95.02631 | 1107420 |
| 904 | 817 216 | 30.06659 | 95.07891 | 1106195 |
| 905 | 819 025 | 30.08322 | 95.13149 | 1104972 |
| 906 | 820 836 | 30.09983 | 95.18403 | 1103753 |
| 907 | 822 649 | 30.11644 | 95.23655 | 1102536 |
| 908 | 824 464 | 30.13304 | 95.28903 | 1101322 |
| 909 | 826 281 | 30.14963 | 95.34149 | 1100110 |
| 910 | 828 100 | 30.16621 | 95.39392 | 1098901 |
| 911 | 829 921 | 30.18278 | 95.44632 | 1097695 |
| 912 | 831 744 | 30.19934 | 95.49869 | 1096491 |
| 913 | 833 569 | 30.21589 | 95.55103 | 1095290 |
| 914 | 835 396 | 30.23243 | 95.60335 | 1094092 |
| 915 | 837 225 | 30.24897 | 95.65563 | 1092896 |
| 916 | 839 056 | 30.26549 | 95.70789 | 1091703 |
| 917 | 840 889 | 30.28201 | 95.76012 | 1090513 |
| 918 | 842 724 | 30.29851 | 95.81232 | 1089325 |
| 919 | 844 561 | 30.31501 | 95.86449 | 1088139 |
| 920 | 846 400 | 30.33150 | 95.91663 | 1086957 |
| 921 | 848 241 | 30.34798 | 95.96874 | 1085776 |
| 922 | 850 084 | 30.36445 | 96.02083 | 1084599 |
| 923 | 851 929 | 30.38092 | 96.07289 | 1083424 |
| 924 | 853 776 | 30.39737 | 96.12492 | 1082251 |
| 925 | 855 625 | 30.41381 | 96.17692 | 1081081 |
| 926 | 857 476 | 30.43025 | 96.22889 | 1079914 |
| 927 | 859 329 | 30.44667 | 96.28084 | 1078749 |
| 928 | 861 184 | 30.46309 | 96.33276 | 1077586 |
| 929 | 863 041 | 30.47950 | 96.38465 | 1076426 |
| 930 | 864 900 | 30.49590 | 96.43651 | 1075269 |
| 931 | 866 761 | 30.51229 | 96.48834 | 1074114 |
| 932 | 868 624 | 30.52868 | 96.54015 | 1072961 |
| 933 | 870 489 | 30.54505 | 96.59193 | 1071811 |
| 934 | 872 356 | 30.56141 | 96.64368 | 1070664 |
| 935 | 874 225 | 30.57777 | 96.69540 | 1069519 |
| 936 | 876 096 | 30.59412 | 96.74709 | 1068376 |
| 937 | 877 969 | 30.61046 | 96.79876 | 1067236 |
| 938 | 879 844 | 30.62679 | 96.85040 | 1066098 |
| 939 | 881 721 | 30.64311 | 96.90201 | 1064963 |
| 940 | 883 600 | 30.65942 | 96.95360 | 1063830 |
| 941 | 885 481 | 30.67572 | 97.00515 | 1062699 |
| 942 | 887 364 | 30.69202 | 97.05668 | 1061571 |
| 943 | 889 249 | 30.70831 | 97.10819 | 1060445 |
| 944 | 891 136 | 30.72458 | 97.15966 | 1059322 |
| 945 | 893 025 | 30.74085 | 97.21111 | 1058201 |
| 946 | 894 916 | 30.75711 | 97.26253 | 1057082 |
| 947 | 896 809 | 30.77337 | 97.31393 | 1055966 |
| 948 | 898 704 | 30.78961 | 97.36529 | 1054852 |
| 949 | 900 601 | 30.80584 | 97.41663 | 1053741 |
| 950 | 902 500 | 30.82207 | 97.46794 | 1052632 |

| $N$ | $N^2$ | $\sqrt{N}$ | $\sqrt{10N}$ | $1/N$ .00 |
|---|---|---|---|---|
| 950 | 902 500 | 30.82207 | 97.46794 | 1052632 |
| 951 | 904 401 | 30.83829 | 97.51923 | 1051525 |
| 952 | 906 304 | 30.85450 | 97.57049 | 1050420 |
| 953 | 908 209 | 30.87070 | 97.62172 | 1049318 |
| 954 | 910 116 | 30.88689 | 97.67292 | 1048218 |
| 955 | 912 025 | 30.90307 | 97.72410 | 1047120 |
| 956 | 913 936 | 30.91925 | 97.77525 | 1046025 |
| 957 | 915 849 | 30.93542 | 97.82638 | 1044932 |
| 958 | 917 764 | 30.95158 | 97.87747 | 1043841 |
| 959 | 919 681 | 30.96773 | 97.92855 | 1042753 |
| 960 | 921 600 | 30.98387 | 97.97959 | 1041667 |
| 961 | 923 521 | 31.00000 | 98.03061 | 1040583 |
| 962 | 925 444 | 31.01612 | 98.08160 | 1039501 |
| 963 | 927 369 | 31.03224 | 98.13256 | 1038422 |
| 964 | 929 296 | 31.04835 | 98.18350 | 1037344 |
| 965 | 931 225 | 31.06445 | 98.23441 | 1036269 |
| 966 | 933 156 | 31.08054 | 98.28530 | 1035197 |
| 967 | 935 089 | 31.09662 | 98.33616 | 1034126 |
| 968 | 937 024 | 31.11270 | 98.38699 | 1033058 |
| 969 | 938 961 | 31.12876 | 98.43780 | 1031992 |
| 970 | 940 900 | 31.14482 | 98.48858 | 1030928 |
| 971 | 942 841 | 31.16087 | 98.53933 | 1029866 |
| 972 | 944 784 | 31.17691 | 98.59006 | 1028807 |
| 973 | 946 729 | 31.19295 | 98.64076 | 1027749 |
| 974 | 948 676 | 31.20897 | 98.69144 | 1026694 |
| 975 | 950 625 | 31.22499 | 98.74209 | 1025641 |
| 976 | 952 576 | 31.24100 | 98.79271 | 1024590 |
| 977 | 954 529 | 31.25700 | 98.84331 | 1023541 |
| 978 | 956 484 | 31.27299 | 98.89388 | 1022495 |
| 979 | 958 441 | 31.28898 | 98.94443 | 1021450 |
| 980 | 960 400 | 31.30495 | 98.99495 | 1020408 |
| 981 | 962 361 | 31.32092 | 99.04544 | 1019368 |
| 982 | 964 324 | 31.33688 | 99.09591 | 1018330 |
| 983 | 966 289 | 31.35283 | 99.14636 | 1017294 |
| 984 | 968 256 | 31.36877 | 99.19677 | 1016260 |
| 985 | 970 225 | 31.38471 | 99.24717 | 1015228 |
| 986 | 972 196 | 31.40064 | 99.29753 | 1014199 |
| 987 | 974 169 | 31.41656 | 99.34787 | 1013171 |
| 988 | 976 144 | 31.43247 | 99.39819 | 1012146 |
| 989 | 978 121 | 31.44837 | 99.44848 | 1011122 |
| 990 | 980 100 | 31.46427 | 99.49874 | 1010101 |
| 991 | 982 081 | 31.48015 | 99.54898 | 1009082 |
| 992 | 984 064 | 31.49603 | 99.59920 | 1008065 |
| 993 | 986 049 | 31.51190 | 99.64939 | 1007049 |
| 994 | 988 036 | 31.52777 | 99.69955 | 1006036 |
| 995 | 990 025 | 31.54362 | 99.74969 | 1005025 |
| 996 | 992 016 | 31.55947 | 99.79980 | 1004016 |
| 997 | 994 009 | 31.57531 | 99.84989 | 1003009 |
| 998 | 996 004 | 31.59114 | 99.89995 | 1002004 |
| 999 | 998 001 | 31.60696 | 99.94999 | 1001001 |
| 1000 | 1 000 000 | 31.62278 | 100.00000 | 1000000 |

# Reference Table to Standard Statistics Textbooks

The 36 sections from SELF-CORRECTING PROBLEMS IN STATISTICS may be used with the chapters of standard statistics textbooks as follows. Sections shown in parentheses contain supplementary materials that build on concepts discussed in the textbook chapter.

| **Textbook Chapter** | Alder and Roessler, *Introduction to Probability and Statistics,* 4th ed. | Bryant, *Statistical Analysis,* 2d ed. | Chao, *Statistics: Methods and Analysis* | Chou, *Applied Business and Economic Statistics* | Clark and Schkade, *Statistical Methods for Business Decisions* | Clelland, et. al., *Basic Statistics With Business Applications* | Croxton, Cowden and Bolch, *Practical Business Statistics,* 4th ed. |
|---|---|---|---|---|---|---|---|
| **1** | | | | | 2 | | |
| **2** | 2 | 5-7 | | 1 | 3 | 2, (1) | 2 |
| **3** | | 15 | 5-7 | | 4 | 5, 6 | 3 |
| **4** | 3, 4 | 2-4, 8, 9 | 2, 3 | | | 7 | 4 |
| **5** | 5 | 11-13, 16-19, 25 | 4 | 2 | | | 4 |
| **6** | 7, (6) | 10, 14, 20, 22 | 8, 9 | 3 | 5, 6 | 3, 4 | 5-7 |
| **7** | 7 | 26, 28 | 7, 10 | 4 | 7 | | 7 |
| **8** | 8, 9 | 23, 24 | 7 | 5, (6) | 7 | 8, 9 | 8-10 |
| **9** | 16-18, 11, (12) | 32-35 | 11, 12, 14 | 7-10 | 8-10, 11, 14 | 10-14 | |
| **10** | 13, 19 | 27 | 16, 20 | 11, 12, 16-18 | 13 | 16-20 | 11, 16, 17 |
| **11** | | | 13, 17-19 | 13, 14, 19, 20 | 16-19 | 26 | 12, 13, 18, 19 |
| **12** | 26-28 | | 21, 22, 25 | | 20 | 26, 28 | 14, 20, 25 |
| **13** | 21, 22 | 15 | 23-25 | | 21, 22 | 32-35 | 30 |
| **14** | 32 | | 26, 28, (27) | 26, 28, (27) | 25 | 27, 36 | 26 |
| **15** | 33-35, (36) | | 32-35, (36) | 32 | 23, 24 | 15, 29 | 28 |
| **16** | 25 | | 15, 29, 30, (31) | | 26, 28 | 30, 31 | 27, 28 |
| **17** | 23, (24) | | 20, 28 | 33 | 27, 28 | | 21-24 |
| **18** | | | | 34, 35 | 27, 28 | | 32 |
| **19** | | | | 36 | 33 | | 33 |
| **20** | | | | | 34-36 | | 34, 35 |
| **21** | | | | | 32 | | 36 |
| **22** | | | | | | | |
| **23** | | | | | | | |
| **24** | | | | | | | |

# Reference Table to Standard Statistics Textbooks

The 36 sections from SELF-CORRECTING PROBLEMS IN STATISTICS may be used with the chapters of standard statistics textbooks as follows. Sections shown in parentheses contain supplementary materials that build on concepts discussed in the textbook chapter.

| Textbook Chapter | Croxton, Cowden and Klein, *Applied General Statistics*, 3d ed. | Dixon and Massey, *Introduction to Statistical Analysis*, 3rd ed. | Freund, *Modern Elementary Statistics*, 3d ed. | Freund and Williams, *Elementary Business Statistics* | Guenther, *Concepts of Statistical Inference* | Hadley, *Introduction to Business Statistics* | Hoel, *Elementary Statistics*, 2d ed. | Huntsberger, *Elements of Statistical Inference*, 2d ed. |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | 5, 6 | 5 | | |
| 2 | | 2 | 2 | 2 | 7 | 2-4, 6-10 | 2-4 | 2 |
| 3 | | 3, 4 | 3 | 3, 4 | 2-4, 8, 9 | 11, 13, 14, (12) | 5 | 3, 4 |
| 4 | | 8, 9 | 4 | 32 | 10, 16, 17, 20 | 16-21, (22) | 6, 7 | 5 |
| 5 | | 7 | 5 | 5, 7 | 11-13, 18, 19, 25 | 15, 30 | 8-10 | 6, 7 |
| 6 | | 11, 13, 16, 17, 19 | 15 | 6, 15 | 14, 20-22 | | 11, 13, 14, (12) | 8-10 |
| 7 | 1 | 10, 14, 20, 25 | 6, 7 | 8-10 | 23, (24) | 26, (27, 28) | 16-20 | 11, 13, 14 |
| 8 | 2 | 12, 13, 18, 19, 25 | 8, 9 | 11, 13, 14, (12), (25) | 26, 28, (27) | 32-36, (1) | 28 | 16, 17, 19, 20 |
| 9 | 3 | | 11-13, 16-19 | 16-22 | | | 26, 27 | 12, 18 |
| 10 | 4 | 23, 24 | 25 | 26, 33-35, (36) | | | 21, 22, 25 | 21, 22 |
| 11 | | 26-28 | 10, 14, 20-22 | | | | | 26, 28, (27) |
| 12 | 33 | | 23, 24 | 23, (24) | | | 23, 24 | 23, 24 |
| 13 | 33 | 14, 20-22 | | | | | 32-35 | |
| 14 | 34 | 16 | 26 | | | | | |
| 15 | 34 | | 28, (27) | | | | | |
| 16 | 35 | | | | | | | |
| 17 | 32 | | | | | | | |
| 18 | 32 | | | | | | | |
| 19 | 26 | | | | | | | |
| 20 | 27 | 5-7 | | | | | | |
| 21 | 27, 28 | | | | | | | |
| 22 | 36 | | | | | | | |
| 23 | 7 | | | | | | | |
| 24 | 8, 9, 11-13, 16-19 | | | | | | | |
| 25 | 10, 14, 20, 22 | | | | | | | |
| 26 | 23-25, 28 | | | | | | | |
| 27 | | | | | | | | |
| 28 | | | | | | | | |

# Reference Table to Standard Statistics Textbooks

The 36 sections from SELF-CORRECTING PROBLEMS IN STATISTICS may be used with the chapters of standard statistics textbooks as follows. Sections shown in parentheses contain supplementary materials that build on concepts discussed in the textbook chapter.

| Textbook Chapter | Jedamus and Frame, *Business Decision Theory* | Kurnow, et. al., *Statistics for Business Decisions* | Mendenhall, *Introduction to Probability and Statistics,* 2d ed. | Neter and Wasserman, *Fundamental Statistics for Business and Economics,* 3d ed. | Ostle, *Statistics in Research,* 2d ed. | Peters and Summers, *Statistical Analysis for Business Decisions* | Richmond, *Statistical Analysis,* 2d ed. | Snedecor and Cochran, *Statistical Methods,* 6th ed. |
|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | 5 | | 2-7 |
| 2 | 15, 30 | 2 | | | | 6 | (1) | 8, 9, 11, 13, 16, 17, 19 |
| 3 | 5-7 | 3, 4, (1) | 2-4 | 1 | 5-7 | 15 | 2, 3 | 21, 25 |
| 4 | 8, 10 | | 5 | | 2-4 | 7 | 4 | 12, 18 |
| 5 | 16, 20 | 5-7 | 6 | 2 | 8-10 | 2-4, 7 | 5-9 | |
| 6 | 15, 29 | 8-10 | 7 | 3, 4 | 11-14, 25 | 8, 9 | 11, 12, 16-18 | 26 |
| 7 | 30 | 11, 14, (13) | 7 | 5-7 | 16-22, 25 | 16-19, 25 | 13, 19 | 28 |
| 8 | 2-4, 6 | 16, 17, 20, (19) | 8-12, 14, 16-18, 20 | 8-10 | 26, 28 | 11, (12, 13) | 10, 14, 20 | 10, 14, 20 |
| 9 | 7, 9, 11-14 | | 13, 19, 25 | 11-14 | 27 | 10, 14, 20 | | 22 |
| 10 | 16-18, 20, (19) | | 26, (27, 28) | | | 29-31 | 15, 30, (29) | 23 |
| 11 | 31 | | 22, (21) | 15 | 23, 24 | 31 | 21, 22 | 24 |
| 12 | 31 | 18, 20, (12) | 23, 24 | 16, 17, 29 | 24 | 21, 22 | 23, 25, (24) | 24 |
| 13 | | 26, 28, (27) | | 17-20 | | 23, 24 | | 27 |
| 14 | | 33-35, (36) | | 21-23, 30, (24) | | 26 | | |
| 15 | | | | 26-28 | | 27 | 33 | 27 |
| 16 | | | | 32 | | 28 | 34 | |
| 17 | | | | 33 | | | 34 | |
| 18 | | | | 35, 36 | | | 35 | |
| 19 | | | | 34 | | | 26-28, (36) | |
| 20 | | | | | | | 32 | |

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| Textbook Chapter | Spurr and Bonini, *Statistical Analysis for Business Decisions* | Stockton, *Introduction to Business and Economic Statistics*, 3d ed. | Walker and Lev, *Statistical Inference* | Wallis and Roberts, *Statistics: A New Approach* | Weinberg and Schumaker, *Statistics: An Intuitive Approach*, 2d ed. | Wonnacott and Wonnacott, *Introductory Statistics* | Yamane, *Statistics, an Introductory Analysis*, 2d ed. |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | | 5-7 | | 3 | 2-4 | 2 |
| 3 | | 2, (1) | 8, 10, 14, 20 | | 4 | 5 | 3 |
| 4 | 1, 2 | | 22, (21) | 8 | 4 | 6, 7 | 4 |
| 5 | 3 | 3 | 2-4 | | 2 | 6 | 5, 6 |
| 6 | 4 | 4 | 8, 9 | 1, 2 | 3, 4 | 8-10 | 7 |
| 7 | 5, 6 | 5, 7, (6) | 11-13, 16-19 | 3 | | 11 | 8-10 |
| 8 | 7 | 8-11, 14, (12-13) | 25 | 4 | 7 | 12-14, 25 | 16, 17, 20 |
| 9 | 15 | 16-22 | 23 | 32 | 8, 9 | 16-20 | 15 |
| 10 | 30, 31 | | 26 | 5, (6) | 9 | 23, 24 | 11 |
| 11 | 8, 9, 11 | (1) | | 7-10 | 10, 16 | 26 | 32 |
| 12 | 12, 16-18 | 33 | | 16 | 17, 20 | 26 | 33 |
| 13 | 10, 13, 14, 19, 20 | 33 | 27, 28 | 17-20, 23 | 11 | 27 | 34, 35 |
| 14 | | 34 | 24 | 11-14 | 13, 19 | 28 | 26 |
| 15 | 30, (29) | 35 | | | 22 | 15, 30 | 28 |
| 16 | 31 | 32 | | | 26 | | 6 |
| 17 | | | | 26, (27, 28) | 28 | | 13, 19, (12, 18) |
| 18 | 32 | (36) | | 33-35 | 26, 28 | | 10, 14, 20 |
| 19 | 33 | 26 | | | | | |
| 20 | 34 | 28 | | | 23 | | 21, 22 |
| 21 | 35 | 27 | | | | | 23-25 |
| 22 | 26 | | | | | | 27 |
| 23 | 27, 28 | | | | | | 36 |
| 24 | 36 | | | | | | |

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